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Bootstrap Percolation and Kinetically Constrained Models in Homogeneous and Random Environments

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כּוֹס הַזְּכוּכִית רַךְ מְאֹד וְרַךְ יֵינוֹ

נִדְמוּ עֲדֵי יֵשׁ בָּם סָפֵק לְכָל עַיִן

דּוֹמֶה כְּאִלּוּ שָׂם יַיִן בְּלִי כּוֹסוֹ וְכֵן

דּוֹמֶה כְּאִלּוּ רֵיק הַכּוֹס בְּלִי יַיִן

Vidal Benveniste, *The Wine Glass*

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Abstract

This thesis concerns with Kinetically Constrained Models and Bootstrap Percolation, two topics in the intersection between probability, combinatorics and statistical mechanics. Kinetically constrained models were introduced by physicists in the 1980's to model the liquid-glass transition [**26, 44**], whose understanding is still one of the big open questions in condensed matter physics. They have been studied extensively in the physics literature in the hope to shed some light on this problem, and in the last decade they have also received an increasing attention in the probability community. We will see that even though they belong to the well established field of interacting particle systems with stochastic dynamics, kinetically constrained models pose challenging and interesting problems requiring the development of new mathematical tools.

Bootstrap percolation, on the other hand, is a class of monotone cellular automata, namely discrete in time and deterministic dynamics, the first example being the r-neighbor bootstrap percolation introduced in 1979 [**21**]. Since then, the study of bootstrap percolation has been an active domain in both the combinatorial and probabilistic communities, with several breakthroughs in the recent years (see [**15, 17**] and references therein).

Though introduced in different contexts, kinetically constrained models and the bootstrap percolation, as we will see, are intimately related; and one may think of bootstrap percolation as a deterministic counterpart of kinetically constrained models, and of kinetically constrained models as the natural stochastic version of bootstrap percolation.

We will focus on the study of time scales for both models, motivated by the attempt to explain the fact that observed time scales in glassy materials are anomalously long.

Keywords: kinetically constrained models, bootstrap percolation, stochastic models in random environments, glassy dynamics, interacting particle systems, hitting times, relaxation time, metastability

Résumé

Cette thèse est consacrée à l'étude des modèles aux contraintes cinétiques et de la percolation bootstrap, dans l'intersection entre les probabilités, la combinatoire et la physique statistique. Les modèles aux contraintes cinétiques ont été introduits dans les années 80 pour modéliser la transition liquide-verre [**26, 44**], dont la compréhension reste toujours un des plus grands problèmes de la physique de la matière condensée. Ils ont été depuis profondément étudiés par des physiciens dans l'espoir d'éclaircir ce problème et la communauté mathématique s'en intéresse de plus en plus lors de la dernière décennie. Ces modèles sont des systèmes de particules en interaction dont la théorie générale est maintenant bien établie. Leur analyse rencontre tout de même des difficultés qui nécessitent le développement de nouveaux outils mathématiques.

La percolation bootstrap est une classe d'automates cellulaires, i.e. déterministes en temps discret. Elle a été considérée pour la première fois en 1979 [**21**] et son étude est depuis devenue un domaine actif en combinatoire et en probabilités.

Les modéles aux contraintes cinétiques et la percolation bootstrap ont été introduits séparément mais sont fortement reliés – on verra que la percolation bootstrap est une version déterministe des modéles aux contraintes cinétiques et que ces derniers sont une version stochastique de la percolation bootstrap.

On se concentrera sur les échelles de temps de ces deux modèles dans le but de comprendre le comportement des matériaux vitreux.

Mots clés : modèles aux contraintes cinétiques, percolation bootstrap, modèles stochastiques en environnement aléatoire, dynamique vitreuse, systèmes de particules en interaction, temps d'atteinte, temps de relaxation, metastabilité

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CHAPTER 1

Introduction

1.1. Glasses and the liquid-glass transition

Glass is a material widely present in our daily life, and yet a complete microscopic understanding of its state of matter and its formation is still out of reach for condensed matter physicists [**10, 11**]. At the heart of this puzzle lies the fact that glasses display properties of both solids and liquids – despite their rigidity, the arrangement of their atoms and molecules is disordered, like that of a liquid. This seems to be a contradiction: if the arrangement of the molecules is amorphous, why would they stay in a fixed position and keep the glass rigid?

The liquid-glass transition occurs when a liquid is cooled below its melting temperature very rapidly, preventing the nucleation of the ordered crystal structure. Roughly speaking, the liquid-crystal transition is avoided since molecules do not have enough time to organize themselves and form the ordered crystal structure. In this way the liquid enters a metastable supercooled phase in which the molecules move slower and slower as the temperature is decreased until they get trapped in the structureless glass state. Even though this state is not thermodynamically stable, relaxation times are out of reach in any reasonable experiment and the system gets stuck in the amorphous solid glass state. Experiments show that the slowing down of dynamics in the vicinity of the liquid-glass transition is extremely sharp – relaxation times can increase by 14 orders of magnitude upon a small decrease in temperature. This dramatic growth of time scales is related to the fact that when the temperature is lowered the density is augmented: molecules tend to obstruct one another and the motion becomes very cooperative, giving rise to large blocked structures. There is indeed a clear coexistence of fast and slow regions, a phenomenon that is called dynamical heterogeneities.

1.2. Kinetically constrained models

One attempt to explain the phenomenology described above relies on the idea that the glass transition is a purely dynamical phenomenon, and the static interactions play a minor role (see [**28, 3, 27**]). This approach leads to the study of kinetically constrained models (KCMs), which is a family of interacting particle systems. These models are Markov processes constructed in such a way, that they have a very simple non-interacting equilibrium, and local kinetic constraints that freeze sites with a dense neighborhood. These two properties imitate the dynamical aspect of the glassy behavior – when static interactions do not play an important role, a coarse grained version of the glass will look like a non-interacting system from the thermodynamic point of view, but small regions will not be able to equilibrate when their local environment is dense. The validity of this framework, that the dynamical effect captures the central mechanism behind the liquid-glass transition, is still under debate; but still, both numerical and theoretical studies confirm that KCMs show indeed a behavior typical of glassy phenomenology, including super-Arrhenius slow down [**26, 48**], non-trivial spatio-temporal fluctuations corresponding to dynamical heterogeneities [**27**], aging phenomena [**48, 24**] and ergodicity breaking transition [**50**].

KCMs live on a (possible infinite) graph G, where each vertex has a state taken from a set $\mathcal{S}$. That is, the state space of the Markov process is $\Omega = \mathcal{S}^G$. We will call the vertices of G *sites*, and an element of Ω will be called *configuration*. The equilibrium is non-interacting, i.e., an independent product measure

$$\mu = \bigotimes_{x\in G} \mu_x,$$

where μ_x, for each site x, is a measure on $\mathcal{S}$. In the models that we will study all these measures are equal. μ_x is a measure on the space of states, thus it operates on functions from $\mathcal{S}$ to $\mathbb{R}$. It will be useful to apply μ_x to functions on the entire configuration space Ω, by fixing the configuration at the sites $G \setminus \{x\}$ and integrating over the state at x. This way we obtain another function on Ω that does not depend on the configuration at x. It could also be thought of as the expected value conditioned on the configuration outside x.

Before defining the KCM, we first consider the unconstrained model – on each site there is a Poisson clock, ringing at rate 1. Whenever the clock rings, the site forgets its state, and chooses a new one at random, according to the equilibrium measure. This description of the process is called the *graphical construction* (see [**38**]). Equivalently, we can describe this dynamics using the generator of the Markov semi-group. The generator is an operator defined on *local functions*, i.e., functions from Ω to $\mathbb{R}$ that depend only on the states of finitely many sites. For the unconstrained dynamics,

$$\mathcal{L}^{\text{unconstrained}} f = \sum_{x\in G} \left(\mu_x f - f\right).$$

Let us now add the constraints – for every site x, we define its constraint c_x. It equals either 0 or 1, and depends on the configuration outside x, i.e., the states of the vertices in $G \setminus \{x\}$. Then the generator of the associated KCM is defined by

$$\mathcal{L} f = \sum_{x\in G} c_x \left(\mu_x f - f\right). \tag{1.2.1}$$

In the graphical construction, we will take the same Poisson clocks with rate 1, but unlike the unconstrained case, we only update the state of a vertex when the constraint is satisfied, i.e., when $c_x = 1$. It is important to stress that once the constraint is satisfied, the new state that we give to the site is chosen randomly according to the measure μ_x, and it does not depend on the configuration of the system. This assures that the model is indeed reversible with respect to the (not necessarily unique) invariant measure μ.

There are many KCMs with different possible G, $\mathcal{S}$, μ and c_x. A common choice for $\mathcal{S}$ is $\{0,1\}$ and $\mu_x = \text{Ber}\,(1-q)$ for some parameter $q \in [0,1]$. These are called *0-1 spin models*, and sites are said to be *empty* or *occupied* if their state is 0 or 1 respectively. In the 0-1 spin models that we discuss, the constraint is a decreasing function of the configuration, that is, more empty sites could only help satisfying the constraint.

In order to have a more concrete example in mind, we will introduce now the Fredrickson-Andersen 2 spin facilitated KCM on $\mathbb{Z}^2$ (FA2f on $\mathbb{Z}^2$), see [**26**]. It is a 0-1 spin model on the graph $G = \mathbb{Z}^2$, with the constraint

$$c_x(\eta) = \begin{cases} 1 & \sum_{y\sim x} (1-\eta_y) \geq 2 \\ 0 & \text{otherwise,} \end{cases}$$

i.e., the constraint is satisfied at x if at least 2 of its neighbors are empty. This is an example of a *homogeneous* KCM, meaning that the constraint is translation invariant.

The dynamics described above, in which sites are resampled at a certain rate, is called Glauber dynamics. For systems in which the total number of occupied sites is conserved we introduce Kinetically Constrained Liquid Gas (KCLG) models. They evolve according to the Kawasaki dynamics, in which sites exchange their state rather than resampling it. In this case the constraint will be defined for edges of the graph rather than vertices. We will study such a model in chapter 5.

1.3. Ergodicity and the bootstrap percolation

From now on we will focus on 0-1 spin KCMs, and assume that the constraint is decreasing in the configuration. The first question we ask about a KCM is whether or not it is ergodic, i.e., if it must equilibrate eventually (in the $L^2(\mu)$ sense). One possible obstacle that could make the system non-ergodic is the existence of a large cluster of occupied sites, each of which has many neighbors inside the cluster, such that none could ever be emptied. In FA2f on $\mathbb{Z}^2$, for example, two infinite consecutive rows of occupied sites will form such a cluster – each site only has one neighbor outside the cluster, so it could never have the two empty neighbors needed in order for the constraint to be satisfied. In order to identify these blocked clusters, we introduce the bootstrap percolation associated with the KCM.

The bootstrap percolation is a deterministic dynamics on Ω in discrete time, where at each time step t empty sites stay empty, and an occupied site x becomes empty if the constraint is satisfied, namely

$$\eta_x(t) = \begin{cases} 1 & \eta_x(t-1) = 1 \text{ and } c_x(\eta(t-1)) = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.1}$$

In other words, whenever the constraint is satisfied we empty the site, and never fill it again. This way, if we wait for long enough, only the blocked clusters will remain occupied. The initial configuration for the bootstrap percolation is random according to the measure μ (which is the equilibrium of the KCM), so it will give us information on the KCM starting from equilibrium. Going back to the example given in the previous paragraph, the blocked cluster described for FA2f in $\mathbb{Z}^2$ has μ-probability 0 to appear in the initial configuration.

The bootstrap percolation is monotone, since sites that become empty could never be filled. Therefore, one can define a limiting configuration at time ∞, saying if a site is emptiable, or forever occupied. Moreover, there exists a critical probability above which all sites are emptiable with probability 1, and below which some sites could remain occupied forever:

$$q_c = \inf \{q \,:\, \mu(\text{all sites are emptiable}) = 1\}.$$

PROPOSITION 1.3.1 (Proposition 2.5 of [**19**]). *The KCM is ergodic if and only if all sites are emptiable for the bootstrap percolation with probability* 1. *In particular, it is ergodic for* $q > q_c$ *and not ergodic for* $q < q_c$.

For the FA2f on $\mathbb{Z}^2$ model introduced above, it is shown in [**51**] that $q_c = 0$, so all sites are emptiable with probability 1 whenever $q > 0$. For $q = 0$, however, all sites are occupied in the initial state, so the constraint is never satisfied, and all sites remain occupied forever.

In order to understand this transition, we can try to study the bootstrap percolation in finite regions.

DEFINITION 1.3.2. Consider a set V of sites and a configuration $\eta \in \Omega$. The span of V for η is the set of sites that are emptiable for the configuration which equals η on V and 1 outside V. If V is contained in its span for η we say that it is internally spanned for η.

The scaling of the probability that the box $[L]^2 \subset \mathbb{Z}^2$ is internally spanned has been studied in [**2, 34**], finding that for small q it becomes very close to 1 when L passes the critical scale $e^{\frac{\pi^2}{18}\frac{1}{q}}$.

REMARK 1.3.3. The notation adopted here it the one usually used in the study of KCMs, which is not necessarily the more common in the bootstrap percolation community. For example, empty and occupied sites are often referred to as "infected" and "healthy" by people studying the bootstrap percolation; and their label (0 or 1) is the opposite of the one defined for KCM. Slightly less confusing differences are that the parameter q is usually called p, and the naming of the constraints is not the same, e.g., FA2f introduced above is called the "2-neighbor bootstrap percolation".

1.4. Time scales

An important aspect of both the bootstrap percolation and KCMs is the divergence of time scale around criticality. For the bootstrap percolation, the most natural time is the time at which a site becomes empty. In the KCM sites are emptied and filled many times, so the relevant scale is when a site becomes empty for the first time (see, e.g., [**19**]). Since the bootstrap percolation empties all possible sites, we can bound from below (with high probability) the first time a site is empty for the KCM by its emptying time for the bootstrap percolation. The other direction, however, is not always true.

If we are interested in understanding the time that it takes for the system to forget its initial state, a good candidate will be the time scale at which correlation is lost – if we take some function $f : \Omega \to \mathbb{R}$ and evaluate it at time 0 and at time t starting from equilibrium, will these two values be (more or less) independent? The time scale on which this correlation decays may depend on f, and taking the worst possible (local) f gives the *relaxation time*. In homogeneous KCMs this scale often describes also the decay of "typical" observables (see [**39**]), but we will see here that in systems that are not homogeneous this will no longer be the case.

Universality results for general homogeneous models in two dimensions have been studied recently for the bootstrap percolation in a series of works that provide a good understanding of their behavior [**6, 15, 17, 33**]. Inspired by the tools developed for the bootstrap percolation, universality results on the KCMs could also be obtained for systems with general translation invariant constraints [**43, 40**].

1.5. Random environments

In the example that we have seen, of FA2f on $\mathbb{Z}^2$, the graph G and the constraints c_x have a very simple form. This describes a rather non-realistic physical system, but still, the common belief is that a big part of the behavior of a system is determined by a few important features. This means that as long as we include these features, a simplified model could still explain observed phenomena. This idea of universality, in the case of KCMs, is supported by the results of [**43, 40**]. They show that as long as the constraints are local and translation invariant, there are just a handful of possible ways in which time scales diverge near criticality.

However, one of the important features that is required for these universality results to hold is the homogeneity of the system; and though many systems are indeed homogeneous, when materials have pollution or defects that appear during their preparation non-homogeneity could play an important role. Such systems are often modeled by a dynamics on a frozen random environment, which, in the case of KCM and bootstrap percolation, will mean that the graph G, the constraints c_x, or the local equilibrium measures μ_x are random.

The bootstrap percolation has been studied in many different random environment, e.g., the polluted lattice [**31, 30**], Galton-Watson trees [**16**], random regular graphs [**9, 35**] and the Erdős-Rényi graph [**36**]. KCMs in random environments have not been studied mathematically (to my knowledge), but they have been treated in the physics literature, see [**45, 52**].

KCMs in random environments contain three types of randomness – the *disorder* (also called *quenched variables*), the initial state, and the evolution of the Markov process. Throughout this thesis we will denote by ν the measure of the disorder (namely the randomness of the environment); and use the letter ω when we want to refer to a specific realization of the quenched variables. The initial state will always be sampled from the equilibrium measure μ and we will use the letter η to represent a configuration of the system. Finally, the randomness of the Markov process will be denoted by $\mathbb{P}_\eta$ and $\mathbb{E}_\eta$ when starting from a given configuration η, or $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ when starting from equilibrium.

1.6. Overview of the results

This monograph covers two main problems – the behavior of time scales for KCMs and bootstrap percolation in random environments (chapters 2,3 and 4), and the relaxation time of the Kob-Andersen model (chapter 5).

We will start with chapter 2, introducing the tools that will be required in the study of KCMs in random environments. KCMs are not attractive – adding empty sites could allow other sites to update and become occupied. This excludes techniques such as monotone coupling and censoring, which are used in the study of attractive Glauber dynamics like the Ising model or the contact process. This is the reason that in the homogeneous case most of the information we have on the time scales of KCMs comes from the relaxation time, which is given by a variational principle, and will be introduced in the first part of chapter 2. However, we will see in chapters 3 and 4 that in non-homogeneous models the relaxation time fails to describe the actual behavior of the system. The reason is that the variational formula defining it focuses on the worst observable, and when the system is not homogeneous it will be determined by the slowest possible regions. The goal of the rest of chapter 2 will be to develop new variational tools that will not share this problem.

In chapter 3 we will apply these tools to several models on $\mathbb{Z}^d$ in which the constraints are random. We will compare different time scales of the KCM and the associated bootstrap percolation, and see that the random environment induces many interesting phenomena. For example, it is possible for the relaxation time to be infinite, whereas the distribution of the emptying time of a site decays exponentially fast. In another model we will see that the divergence of the emptying time of a site is polynomial with a random power that may depend on the environment.

Chapter 4 will discuss KCMs and bootstrap percolation on a random graph, the Galton-Watson tree. Already for the bootstrap percolation we will see an intriguing behavior, that for different types of Galton-Watson trees the proportion of empty sites could decrease in many strange ways, staying almost fixed for long time intervals (that diverge when approaching q_c), and jumping down rapidly between them.

Chapter 5 presents an analysis of the relaxation time of the Kob-Andersen model. This model is homogeneous, but unlike the models discussed before it is of Kawasaki type – particles are not created and destroyed, but jump from one site to the other. We prove a diffusive scaling – in a box of side L the relaxation time is proportional to L^2, whereas previous studies only showed a scaling of $L^2 (\log L)^4$ **[20]**.

1.7. List of conventions and notation

To make the reading of this thesis easier, I list here the notation that have been introduced above and will be used in the following.

- G – the graph on which the model lives. Sometimes we use G also for the vertices of the graph.
- $(\mathcal{S}, \mu_0)$ – the space of values that a site could have and its equilibrium probability. In most cases $\mathcal{S} = \{0, 1\}$ and $\mu_0 = \text{Ber}(1-q)$.
- (Ω, μ) – the state space of the Markov process and its equilibrium probability. $\Omega = \mathcal{S}^G$ and $\mu = \bigotimes_{x\in G} \mu_x$.
- η – a generic configuration (element of Ω). $\eta_x(t)$ is the occupation value of the site x at time t, and η^x is the configuration obtained from η by flipping the site x.
  Warning: we will sometimes need to refer to a sequence of configurations. In these cases we will denote the sequence by $\eta_1, \eta_2, \ldots$, and the occupation of a site x by, e.g., $(\eta_1)_x$.
- c_x – the constraint at the site x.
- $\mathcal{L}$ – the generator of the Markov process (see equation (1.2.1)).
- $\mathbb{P}, \mathbb{E}$ – the measure of the Markov process. We use $\mathbb{P}_\eta$ and $\mathbb{E}_\eta$ when starting from a given configuration η, or $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ when starting from equilibrium.
- ν – the measure of the disorder.
- ω – generic realization of the disorder. The law of ω is ν.
- $\mu_V f$ – expectation of f with respect to the occupation values in $V \subseteq G$.
- $\nabla_x f(\eta) = f(\eta^x) - f(\eta)$.

CHAPTER 2

Variational tools

This chapter will introduce the variational tools that will be used afterwards in the analysis of KCMs. In the first section we define the spectral gap and see some basic properties related to it. More details could be found, for example, in [**37**]. The second section describes the tools presented in [**47**] in order to study hitting times of reversible Markov processes, and in particular KCMs. Finally, the third section concerns with the time spent in a certain event before hitting another event.

2.1. Spectral gap and relaxation time

The Markov processes that we study will be reversible, i.e., their equilibrium dynamics is the same as the time reversed dynamics. To be more precise, consider a general Markov process on a state space Ω with generator

$$\mathcal{L}f = \sum_{\eta,\eta'\in\Omega} L\left(\eta',\eta\right)\left(f\left(\eta'\right)-f\left(\eta\right)\right),$$

i.e., when at state η, the process jumps to η' at rate $L\left(\eta',\eta\right)$. For finite Ω, choosing the state η according to the measure μ, we define the flow of probability from η to η' as

$$R\left(\eta',\eta\right) = L\left(\eta',\eta\right)\mu\left(\eta\right). \tag{2.1.1}$$

The measure μ is called an equilibrium measure, and the process is said to be reversible if for all $\eta,\eta'\in\Omega$

$$R\left(\eta',\eta\right) = R\left(\eta,\eta'\right). \tag{2.1.2}$$

KCMs (see equation (1.2.1)) have this property. In fact, the flow could only be non-zero when $\eta'=\eta^x$ for some site x, and in this case

$$R\left(\eta^x,\eta\right) = c_x(\eta)\,q(1-q)\ \left(\mu(\eta)+\mu(\eta^x)\right), \tag{2.1.3}$$

which shows that R is indeed symmetric.

Equation (2.1.2) is a different way of saying that $\mathcal{L}$ is a self-adjoint operator in the space $L^2\left(\Omega,\mu\right)$, and defining reversibility this way makes sense also for infinite Ω. Self-adjointness allows us to study the process via the spectrum or $\mathcal{L}$. In particular, we will be interested in the second biggest eigenvalue of $\mathcal{L}$, which is called (up to sign) the *spectral gap*.

Before discussing the spectral gap we start by defining the Dirichlet form. Let f be in the domain of $\mathcal{L}$, then the Dirichlet form applied to f is defined as

$$\mathcal{D}f = -\mu\left(f\mathcal{L}f\right). \tag{2.1.4}$$

For the models we will consider (defined in equation (1.2.1)), the following identities will also be useful:

$$\mathcal{D}f = \mu\left(\sum_{x\in G} c_x \mathrm{Var}_x f\right) = \mu\left(\sum_{x\in G} c_x q(1-q)\left(\nabla_x f\right)^2\right) = \frac{1}{2}\sum_{\eta\in\Omega}\sum_{x\in G} R\left(\eta^x,\eta\right)\left(\nabla_x f\right)^2. \tag{2.1.5}$$

We are now ready to define the spectral gap and the relaxation time

$$\mathrm{gap}\left(\mathcal{L}\right) = \sup_{f:\mathrm{Var}f\neq 0}\frac{\mathcal{D}f}{\mathrm{Var}f}, \tag{2.1.6}$$

$$\tau_{\mathrm{rel}} = \frac{1}{\mathrm{gap}\left(\mathcal{L}\right)}. \tag{2.1.7}$$

The spectral gap will give us the time decay of correlation when starting from equilibrium: for every local f, g and every $t \geq 0$

$$\left|\mathbb{E}_\mu\left[f\left(\eta\left(t\right)\right)g\left(\eta\left(0\right)\right)\right] - \mu\left(f\right)\mu\left(g\right)\right| \leq \mathrm{Cov}(f,g)\, e^{-t/\tau_{\mathrm{rel}}}.$$

In particular, if the process is not ergodic some correlations are never lost, and the spectral gap must be 0. The contrary is not true, i.e., ergodic processes may have 0 gap, as long as the only eigenfunctions of $\mathcal{L}$ with eigenvalue 0 are the constant functions (see, e.g., [**38**]).

2.2. Hitting times

The spectral gap that was introduced in the previous section gives a times scale which describes the relaxation of all observables uniformly. That is, it will be as long as the time that the slowest possible observable takes to relax. In disordered systems, as we will see in more details in chapters 3 and 4, this slowest observable is often much slower than typical times that we are interested in. In these cases, the spectral gap fails to provide a correct description of actual physical time scales of the system, and we must study specific times that could be observed. One of these is the time it takes for the origin to become empty, and more generally hitting times of different events.

We will start by considering a reversible Markov process on a state space Ω with generator $\mathcal{L}$ and equilibrium measure μ.

Definition 2.2.1. Consider an event $A \subseteq \Omega$. The hitting time at A is defined as

$$\tau_A = \inf\left\{t\,:\,\eta(t)\in A\right\}.$$

The hitting time is defined for both the KCM and the bootstrap percolation. For the time it takes to empty the origin we will use the notation

$$\tau_0 = \tau_{\{\eta_0=0\}}. \tag{2.2.1}$$

In the case of KCMs, the hitting time is a random variable that depends on both the initial configuration $\eta \in \Omega$ and the randomness of the Markov process. With some abuse of notation, we will use $\tau_A(\eta)$ and $\tau_0(\eta)$ for the expected value of the hitting time starting at η, i.e.,

$$\tau_A(\eta) = \mathbb{E}_\eta \left[\tau_A\right]. \tag{2.2.2}$$

The function τ_A, for some event $A \subseteq \Omega$, satisfies the following Poisson problem (see, e.g., [**18**, equation (7.2.45)]):

$$\begin{aligned} \mathcal{L}\tau_A &= -1 \text{ on } A^c, \\ \tau_A &= 0 \text{ on } A. \end{aligned} \tag{2.2.3}$$

By multiplying both sides of the equation by τ_A and integrating with respect to μ, we obtain

Corollary 2.2.2. $\mu\left(\tau_A\right) = \mathcal{D}\left(\tau_A\right)$.

Rewriting this corollary as $\mu\left(\tau_A\right) = \frac{\mu(\tau_A)^2}{\mathcal{D}\tau_A}$, it resembles a variational principle introduced in [**5**] that will be useful in the following. In order to formulate it we will need to introduce some notation.

Definition 2.2.3. For an event $A \subseteq \Omega$, V_A is the set of all functions in the domain of $\mathcal{L}$ that vanish on the event A. Note that, in particular, $\tau_A \in V_A$.

Definition 2.2.4. For an event $A \subseteq \Omega$,

$$\overline{\tau}_A = \sup_{0 \neq f \in V_A} \frac{\mu\left(f^2\right)}{\mathcal{D}f}.$$

The following proposition is given in the first equation of the proof of Theorem 2 in [**5**]:

Proposition 2.2.5. $\mathbb{P}_\mu\left[\tau_A > t\right] \leq e^{-t/\overline{\tau}_A}$.

Remark 2.2.6. In particular, Proposition 2.2.5 implies that $\mu\left(\tau_A\right) \leq \bar{\tau}_A$. This, however, could be derived much more simply from Corollary 2.2.2 –

$$\mu\left(\tau_A\right)^2 \leq \mu\left(\tau_A^2\right) \leq \overline{\tau}_A \mathcal{D}\tau_A = \overline{\tau}_A \mu\left(\tau_A\right).$$

Note that whenever τ_A is not constant on A^c this inequality is strict. Thus on one hand Proposition 2.2.5 gives an exponential decay of $\mathbb{P}_\mu\left[\tau_A > t\right]$, which is stronger than the information on the expected value we can obtain from the Poisson problem in equation (2.2.3). On the other hand, $\overline{\tau}_A$ could be longer than the actual expectation of τ_A.

In order to bound the hitting time from below we will formulate a variational principle that characterizes τ_A.

Definition 2.2.7. For $f \in V_A$, let

$$\mathcal{T}f = 2\mu(f) - \mathcal{D}f.$$

Proposition 2.2.8. *τ_A maximizes $\mathcal{T}$ in V_A. Moreover, $\mu(\tau_A) = \sup_{f\in V_A} \mathcal{T} f$.*

Proof. Consider $f \in V_A$, and let $\delta = f - \tau_A$. Using the self-adjointness of $\mathcal{L}$, equation (2.2.3), and the fact that $\delta \in V_A$ we obtain

$$\begin{aligned}\mathcal{T} f &= \mathcal{T}(\tau_A + \delta)\\ &= 2\mu(\tau_A) + 2\mu(\delta) - \mathcal{D}\tau_A - \mathcal{D}\delta + 2\mu(\delta\mathcal{L}\tau_A)\\ &= \mathcal{T}\tau_A - \mathcal{D}\delta.\end{aligned}$$

By the positivity of the Dirichlet form, $\mathcal{T}$ is indeed maximized by τ_A. Finally, by Corollary 2.2.2,

$$\sup_{f\in V_A} \mathcal{T} f = \mathcal{T}\tau_A = 2\mu(\tau_A) - \mathcal{D}\tau_A = \mu(\tau_A).$$

□

As an immediate consequence we can deduce the monotonicity of the expected hitting time:

Corollary 2.2.9. *Let $\mathcal{D}$ and $\mathcal{D}'$ be the Dirichlet forms of two reversible Markov processes defined on the same space Ω, and sharing the same equilibrium measure μ. We denote the expectations with respect to these processes starting at equilibrium by $\mathbb{E}_\mu$ and $\mathbb{E}'_\mu$. Assume that the domain of $\mathcal{D}$ is contained in the domain of $\mathcal{D}'$, and that for every $f \in Dom\mathcal{D}$*

$$\mathcal{D} f \geq \mathcal{D}' f.$$

Then, for an event $A \subseteq \Omega$,

$$\mathbb{E}_\mu\tau_A \leq \mathbb{E}'_\mu\tau_A.$$

We will now restrict ourselves to 0-1 kinetically constrained spin models, i.e., $\mathcal{S} = \{0, 1\}$.

Fix a subgraph H of G, and denote the complement of H in G by H^c. We will compare the dynamics of this KCM to the dynamics restricted to H, with boundary conditions that are the most constrained ones.

Definition 2.2.10. The restricted dynamics on H is the KCM defined by the constraints

$$c_x^H(\eta) = c_x(\eta^H),$$

where, for $\eta \in \{0, 1\}^H$, η^H is the configuration given by

$$\eta^H(x) = \begin{cases}\eta_x & x \in H,\\ 1 & x \in H^c.\end{cases}$$

We will denote the corresponding generator by $\mathcal{L}_H$ and its Dirichlet form by $\mathcal{D}_H$.

Claim 2.2.11. For any f in the domain of $\mathcal{L}$

$$\mathcal{D} f \geq \mu_{H^c}\mathcal{D}_H f.$$

PROOF. $c_x^H \leq c_x$ and $\text{Var}_x f$ is positive, therefore

$$\mathcal{D}f = \mu\left(\sum_x c_x \text{Var}_x f\right) \geq \mu\left(\sum_{x\in H} c_x^H \text{Var}_x f\right).$$

□

The next claim will allow us to relate the spectral gap of the restricted dynamics to the expected hitting time and the persistence function using the variational principles discussed above. For the case $H = G$ the result has been noted in [**40**], but not fully proven.

CLAIM 2.2.12. Let γ_H be the spectral gap of $\mathcal{L}_H$, and fix an event A that depends only on the occupation of the vertices of H. Then for all $f \in V_A$

(1) $\mathcal{D}f \geq \mu(A)\gamma_H\,(\mu f)^2$,
(2) $\mathcal{D}f \geq \frac{\mu(A)}{1+\mu(A)}\gamma_H\,\mu\left(f^2\right)$

PROOF. First, note that $\mu_H(A) \leq \mu_H(f = 0) \leq \mu_H\left(|f - \mu_H f| \geq \mu_H f\right)$. Therefore, by Chebyshev inequality and the fact that $\mu(A) = \mu_H(A)$,

$$\mu(A) \leq \frac{\text{Var}_H f}{(\mu_H f)^2}. \tag{2.2.4}$$

Then, Claim 2.2.11 implies

$$\mathcal{D}f \geq \mu_{H^c}\mathcal{D}_H f \geq \gamma_H \mu_{H^c}\text{Var}_H f \geq \mu\left(A\right)\gamma_H\,\mu_{H^c}\left(\mu_H f\right)^2 \geq \mu\left(A\right)\gamma_H\,\left(\mu f\right)^2$$

by Jensen inequality. For the second part, we use inequality 2.2.4

$$\text{Var}_H f \geq \mu(A)\left(\mu_H(f^2) - \text{Var}_H f\right),$$

which implies

$$\text{Var}_H f \geq \frac{\mu(A)}{1+\mu(A)}\mu_H(f^2).$$

The result then follows by applying Claim 2.2.11. □

2.3. The time spent in E

In this section we consider a reversible Markov process on a state space Ω with generator $\mathcal{L}$ and equilibrium measure μ. Fix an event $E \in \Omega$ and $t > 0$. We will define the time spent in E by time t as

$$T_t^E = \int_0^t \mathbb{1}_E\left(\eta\left(s\right)\right)\mathrm{d}s. \tag{2.3.1}$$

Similarly to the case of the hitting time, this is a random variable that depends on the initial configuration and on the evolution of the Markov process, and we will define again (in analogy with equation (2.2.2)) the averaged version

$$T_t^E\left(\eta\right) = \mathbb{E}_\eta\left[T_t^E\right]. \tag{2.3.2}$$

DEFINITION 2.3.1. Let $E, A \subseteq \Omega$ be two events. The time spent in E before hitting A is

$$T_A^E = T_{\tau_A}^E.$$

Also for T_A^E we define

$$T_A^E(\eta) = \mathbb{E}_\eta\left[T_A^E\right]. \tag{2.3.3}$$

This function solves the Poisson problem (see, e.g., [**18**, equation (7.2.45)])

$$\begin{array}{rcll} \mathcal{L}T_A^E(\eta) & = & -\mathbb{1}_E(\eta) & \eta \notin A, \\ T_A^E(\eta) & = & 0 & \eta \in A. \end{array} \tag{2.3.4}$$

Multiplying both sides by T_A^E and integrating with respect to μ gives

COROLLARY 2.3.2. $\mu\left(T_A^E \mathbb{1}_E\right) = \mathcal{D}\left(T_A^E\right)$.

To finish this section we present the analog of Proposition 2.2.8:

PROPOSITION 2.3.3. *Let $\mathcal{T}_E f = 2\mu(f\mathbb{1}_E) - \mathcal{D}(f)$, and recall Definition 2.2.3. Then T_A^E maximizes $\mathcal{T}_E$ in V_A. Moreover, $\mu\left(T_A^E\mathbb{1}_E\right) = \sup_{f\in V_A} \mathcal{T}_E f$.*

PROOF. Consider $f \in V_A$, and let $\delta = f - T_A^E$. Using the self-adjointness of $\mathcal{L}$, equation (2.3.4), and the fact that $\delta \in V_A$ we obtain

$$\begin{aligned} \mathcal{T}_E f &= \mathcal{T}_E\left(T_A^E + \delta\right) \\ &= 2\mu\left(T_A^E\mathbb{1}_E\right) + 2\mu\left(\delta\mathbb{1}_E\right) - \mathcal{D}T_A^E - \mathcal{D}\delta + 2\mu\left(\delta\,\mathcal{L}T_A^E\right) \\ &= \mathcal{T}_E\tau_A - \mathcal{D}\delta. \end{aligned}$$

By the positivity of the Dirichlet form, $\mathcal{T}_E$ is indeed maximized by τ_A. Finally, by Corollary 2.3.2,

$$\sup_{f\in V_A} \mathcal{T}_E f = \mathcal{T}T_A^E = 2\mu\left(T_A^E\mathbb{1}_E\right) - \mathcal{D}T_A^E = \mu\left(T_A^E\mathbb{1}_E\right).$$

□

CHAPTER 3

Random constraints on $\mathbb{Z}^d$

This chapter will focus on models in random environments that live on the lattice (mostly $\mathbb{Z}^2$). Sections 3.2, 3.3, 3.4 and 3.5 present, essentially, the results of [**47**]. However, while in [**47**] some of the results were simplified in order to make them easier to read, the treatment here will be more complete. One more difference is the proof of section 3.4, which here uses a technique introduced in [**40**] rather than repeating the two-dimensional argument with the (much) more complicated combinatorics of the bootstrap percolation in higher dimensions. The results of sections 3.6 and 3.7 appear here for the first time.

3.1. Introduction of the models

The first model we analyze is the mixed threshold bootstrap percolation and KCM. The graph G is $\mathbb{Z}^d$, and the disorder ω in an element of $\{\text{easy}, \text{difficult}\}^{\mathbb{Z}^d}$. Its measure will depend on a parameter $\pi \in (0,1)$, giving probability π for a site to be difficult and $1-\pi$ to be easy. For an easy site x we define the constraint

$$c_x(\eta) = \begin{cases} 1 & x \text{ has at least one empty neighbor} \\ 0 & \text{otherwise,} \end{cases} \tag{3.1.1}$$

and when x is difficult

$$c_x(\eta) = \begin{cases} 1 & x \text{ has at least } d \text{ empty neighbors} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.2}$$

Note that a more general model would be to allow ω_x to take values in $\{1, \dots, d\}$ and require ω_x empty neighbors, but the result will not change and the notation will become a bit more complicated.

Section 3.5 will concern with a mixture of the North-East model and the FA1f model on the graph $G = \mathbb{Z}^2$. The disorder ω and its measure ν will be the same as for the mixed threshold models. For an easy site x the constraint is given by

$$c_x(\eta) = \begin{cases} 1 & x \text{ is easy and has at least one empty neighbor} \\ 0 & \text{otherwise,} \end{cases} \tag{3.1.3}$$

and for a difficult site

$$c_x(\eta) = \begin{cases} 1 & \eta_{x+e_1} = 0 \text{ and } \eta_{x+e_2} = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.4}$$

In sections 3.6 and 3.7 the graph will be again $\mathbb{Z}^2$, and the disorder ω will give one of two values to each site, which we will call *absent* or *present*. Its measure will depend on a parameter $\pi \in (0,1)$, giving probability π for a site to be absent and $1-\pi$ to be present. The constraints are given by

$$c_x(\eta) = \begin{cases} 1 & x \text{ is present and has at least one empty present neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{3.1.5}$$

for the FA1f model (section 3.6), and

$$c_x(\eta) = \begin{cases} 1 & x \text{ is present and has at least two empty present neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{3.1.6}$$

for the FA2f model (section 3.7).

3.2. Bootstrap percolation on $\mathbb{Z}^2$ with threshold 1 or 2

Consider the bootstrap percolation on $\mathbb{Z}^2$ defined by the constraints in equations (3.1.1) and (3.1.2). We will show that the hitting time τ_0 scales like $q^{-1/2}$ when q is small; and see how the coefficient scales with π when π is also small (but still much bigger than q).

THEOREM 3.2.1. *Consider the bootstrap percolation (see equation* (1.3.1)*) with constraints defined in equations (3.1.1) and (3.1.2) for* $d=2$.

(1) ν*-almost surely there exists* $t_0 = t_0(\omega)$ *such that*

$$\mu\left[\tau_0 \geq t_0 + \frac{e^{\gamma/\pi}}{\pi^3\sqrt{q}}a\right] \leq e^{-\lambda a^2}. \tag{3.2.1}$$

γ *and* λ *are explicit constants.*

(2) ν*-almost surely there exists* $a_0 = a_0(\omega)$ *such that*

$$\mu\left[\tau_0 \leq \frac{e^{\gamma'/\pi}}{2\sqrt{q}}a\right] \leq C_1 q + C_2 a^2 \tag{3.2.2}$$

for all $a \in \left[a_0\sqrt{q}, 1\right]$. γ' *and* C_2 *are explicit constants, and* C_1 *may depend on* π *(but not on* ω, η, *or* q*).*

REMARK 3.2.2. This theorem gives a way to interpolate between the FA1f and FA2f bootstrap percolation – when $\pi = 1$ it is the FA1f model, and τ_0 is indeed of the scale $q^{-1/2}$. As π decreases the coefficient in front of $q^{-1/2}$ increases, bringing us closer to the time scale of the FA2f. If we believe that π could go all the way down to order q (even though the theorem only allows to obtain the scaling for $q \to 0$ after fixing π), we reach the time scale $e^{\gamma/q}$, as in the FA2f model.

REMARK 3.2.3. The proof of Theorem 3.2.1 is essentially that of **[2]**, with the appropriate adaptations. Also in their proof the two exponents γ and γ' are different, but in **[34]** a sharper bound gives two equal exponents. It is thus natural to conjecture that also in our case $\gamma = \gamma'$, and perhaps using some of the methods in **[34]** could show that.

3.2.1. Upper bound (proof of equation (3.2.1)**).** In order to find the upper bound we will construct an explicit path to empty the origin.

DEFINITION 3.2.4. A square (that is, a subset of $\mathbb{Z}^2$ of the form $x + [L]^2$) is *good* if it contains at least one easy site in each line and in each column.

We start by a simple estimation of probability that a square is good.

CLAIM 3.2.5. The probability that a square of size L is good is at least $1 - 2Le^{-\pi L}$.

PROOF.

$$\nu\left[\text{easy site in each line}\right] = \left[1 - (1-\pi)^L\right]^L \geq 1 - Le^{-\pi L}.$$

The same bound holds for ν[easy site in each column], and then we conclude by the union bound. □

DEFINITION 3.2.6. The square $[L]^2$ is *excellent* if for every $2 \leq i \leq L$ at least one of the sites in $\{i\} \times [i-1]$ is easy, and at least one of the sites in $[i-1] \times \{i\}$ is easy. For other squares of side L being excellent is defined by translation.

We will use p_L to denote the probability that a square of side L is excellent.

CLAIM 3.2.7. $p_L \geq e^{-2\gamma/\pi}$, uniformly in L.

PROOF. The probability that the condition holds at step i is $\left(1 - (1-\pi)^{i-1}\right)^2$. We then bound the product $\prod_{i=2}^{L}\left(1 - (1-\pi)^{i-1}\right)$ by its value at $L = \infty$. This could be evaluated by

$$\sum_{i=2}^{\infty} \log\left(1 - (1-\pi)^{i-1}\right) \geq \int_0^{\infty} \mathrm{d}t \, \log\left(1 - (1-\pi)^t\right) = -\frac{1}{\log(1-\pi)} \int_0^{\infty} \mathrm{d}s \, \log\left(1 - e^{-s}\right).$$

The integral $\int_0^\infty \mathrm{d}s \, \log(1 - e^{-s})$ converges, so defining $\gamma = -\int_0^\infty \mathrm{d}s \, \log(1-e^{-s})$ gives the result. Though not essential for the proof, one may note that γ could be calculated explicitly:

$$-\int_0^{\infty} \mathrm{d}s \, \log\left(1 - e^{-s}\right) = \int_0^{\infty} \mathrm{d}s \sum_{k=1}^{\infty} \frac{1}{k} e^{-ks} = \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} \mathrm{d}s \, e^{-ks} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2).$$

□

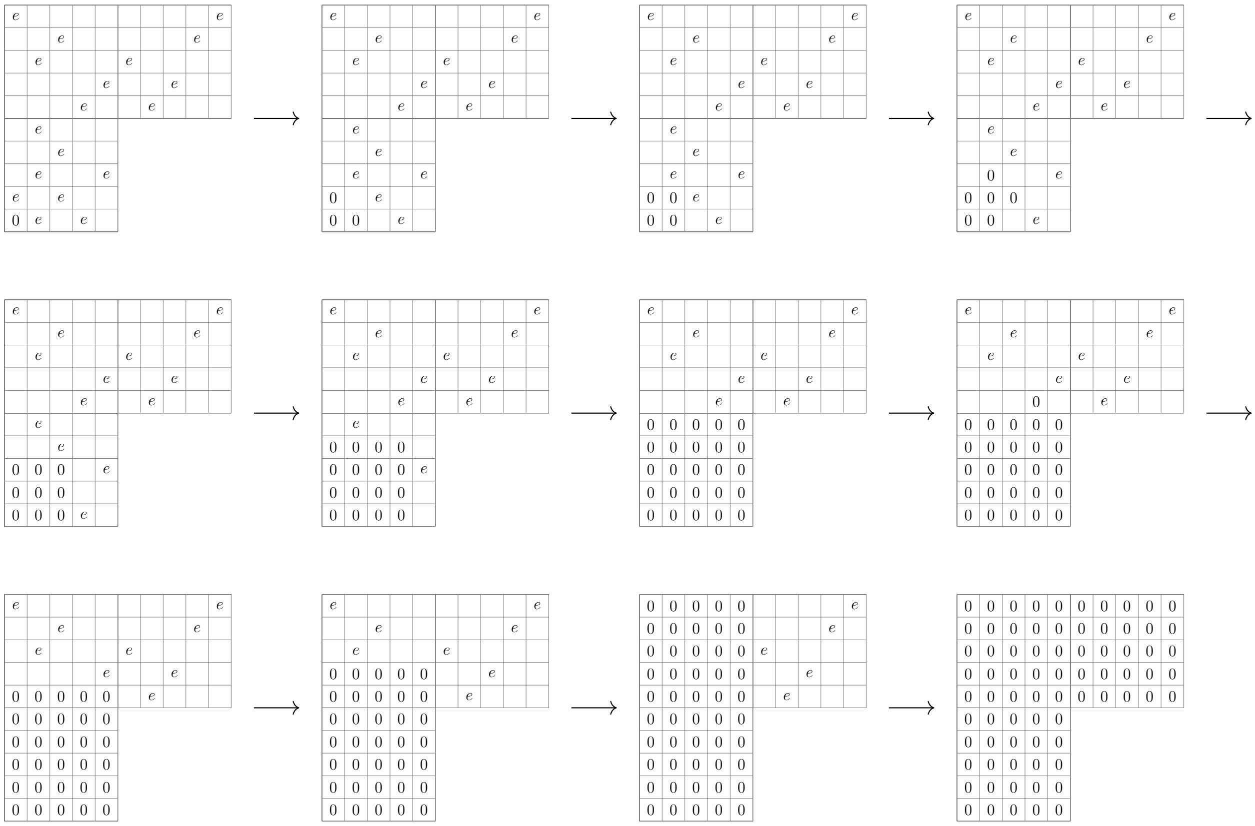

FIGURE 3.2.1. Illustration of claims 3.2.8 and 3.2.9. 0 represents an empty site, otherwise the state is the initial one. e stands for an easy site.

The next two claims will show how a cluster of empty sites could propagate. See figure 3.2.1.

CLAIM 3.2.8. Assume that $[L]^2$ is excellent, and that $(1,1)$ is initially empty. Then $[L]^2$ will be entirely emptied by time L^2.

PROOF. This could be done by induction on the size of the empty square – assume that $[l]^2$ is entirely emptied for some $l \leq L$. By the definition of an excellent square, there is an easy site $x \in \{l+1\} \times [l]$. Its neighbor to the left is empty (since it is in $[l]^2$), so at the next time step this site will also be empty. Once x is empty, the two sites $x \pm e_2$ could be emptied, and then the sites $x \pm 2e_2$ and so on, as long as they stay in $\{l+1\} \times [l]$. Thus, at time l all sites in $\{l+1\} \times [l]$ will be empty, and by the same reasoning the sites of $[l] \times \{l+1\}$ will also be empty. Since $(l+1, l+1)$ has two empty neighbors it will be emptied at step $l+1$, and thus $[l+1]^2$ will be emptied. □

CLAIM 3.2.9. Assume that $[L]^2$ is good, and that it has a neighboring square that is entirely empty by time T. Then $[L]^2$ will be entirely empty by time $T + L^2$.

PROOF. We can empty $[L]^2$ line by line (or column by column, depending on whether its empty neighbor is in the horizontal direction or the vertical one). For each line, we start by emptying the easy site that it contains, and then continue to propagate. □

DEFINITION 3.2.10. Until the end of the proof of the upper bound, L will be the minimal length for which the probability to be good exceeds the critical probability for Bernoulli site percolation by at least 0.01. By Claim 3.2.5 this L exists, and is of order $\frac{\log 1/\pi}{\pi}$ when π is small.

DEFINITION 3.2.11. $\mathcal{C}$ will denote the infinite cluster of good boxes of the form $Li+[L]^2$ for $i \in \mathbb{Z}^2$. $\mathcal{C}_0$ will denote the cluster of the origin surrounded by a path in $\mathcal{C}$, or just the origin if it is in $\mathcal{C}$. $\partial\mathcal{C}_0$ will be the outer boundary of $\mathcal{C}_0$ (namely the boxes out of $\mathcal{C}$ that have a neighbor in $\mathcal{C}_0$). Note that $\mathcal{C}_0$ is finite and that $\partial\mathcal{C}_0$ is connected.

CLAIM 3.2.12. Assume that at time t one of the boxes on $\partial\mathcal{C}_0$ is entirely empty. Then by time $t+t_1$ the origin will be empty, where $t_1 = (|\partial\mathcal{C}_0| + |\mathcal{C}_0|)\, L^2$.

PROOF. By Claim 3.2.9, the boundary $\partial\mathcal{C}_0$ will be emptied by time $t_1 + L^2\,|\partial\mathcal{C}_0|$. Then, at each time step at least one site of $\mathcal{C}_0$ must be emptied, since no finite region could stay occupied forever. □

CLAIM 3.2.13. Assume that a box $Li+[L]^2$ in $\mathcal{C}$ is empty at time t. Also, assume that the graph distance in $\mathcal{C}$ between this box and $\partial\mathcal{C}_0$ is l. Then by time $t+lL^2+t_1$ the origin will be empty.

PROOF. This is again a direct application of claims 3.2.9 and 3.2.12. □

Finally, we will use the following result from percolation theory:

CLAIM 3.2.14. ν-almost surely there exists $l_1 = l_1(\omega)$, such that for $l \geq l_1$ the number of boxes in $\mathcal{C}$ that are at graph distance in $\mathcal{C}$ at most l from $\partial\mathcal{C}_0$ is greater than θl^2. θ is a universal constant (depending only on the choice 0.01 made in Definition 3.2.10).

PROOF. By ergodicity the cluster $\mathcal{C}$ has an almost sure positive density, so in particular

$$\liminf_{l\to\infty} \frac{\left|\mathcal{C}\cap[-l,l]^2\right|}{\left|[-l,l]^2\right|} > 0.$$

By [4], there exists a positive constant ρ such that boxes of graph distance l from the origin must be in the box $\left[-\frac{1}{\rho}l, \frac{1}{\rho}l\right]^2$ for l large enough. Combining these two facts proves the claim. □

This claim together with a large deviation estimate yields

COROLLARY 3.2.15. *ν-almost surely there exists $l_2 = l_2(\omega)$, such that for For $l \geq l_2$, the number of excellent boxes in $\mathcal{C}$ that are at graph distance in $\mathcal{C}$ at most l from $\partial\mathcal{C}_0$ is greater than $\theta' l^2$, where $\theta' = 0.99\,\theta p_L$.*

We can now put all the ingredients together and obtain the upper bound.

Let $l = \frac{e^{\gamma/\pi}}{\sqrt{q}}\,\frac{1}{L^2\pi^3}\,a + l_2$. By Corollary 3.2.15, there are at least $\theta' l^2 \geq \frac{e^{2\gamma/\pi}}{q}\,\frac{1}{L^2\pi^3}\,a^2$ excellent boxes in $\mathcal{C}$ at graph distance in $\mathcal{C}$ at most l from $\partial\mathcal{C}_0$. If one of them contains an empty site at its bottom left corner, the origin will be emptied by time $(l+1)\,L^2 + t_1$, so in this case

$\tau_0 < \frac{e^{\gamma/\pi}}{\pi^3\sqrt{q}}\, a + t_0$, taking $t_0 = t_1 + (l_2+1)\, L^2$. We are left to bound the probability that none of the $\theta' \, \frac{e^{2\gamma/\pi}}{q} \, \frac{1}{L^2}\, a^2$ excellent boxes has an empty corner:

$$(1-q)^{\theta' \, \frac{e^{2\gamma/\pi}}{q} \, \frac{1}{L^2\pi^3}\, a^2} \leq e^{-0.99\,\theta p_L\, e^{2\gamma/\pi}\, a^2} \leq e^{-0.99\theta\, a^2}.$$

□

3.2.2. Lower bound (proof of equation (3.2.2)).

DEFINITION 3.2.16. A rectangle R (that is, a subset of $\mathbb{Z}^2$ of the form $x + [L_1] \times [L_2]$) is pre-internally spanned if there is a vertex $x \in R$ such that starting from entirely filled R, except for one empty site at x, R is internally spanned.

We now fix $k = \frac{B}{\pi}$, for some constant B that will be determined later on.

CLAIM 3.2.17. Fix the rectangle $[a] \times [b]$, for $a \in [k, 2k+2]$ and $b \leq a$. The probability that it is pre-internally spanned is at most $e^{-\gamma_1/\pi}$.

PROOF. In order to prevent the rectangle from being pre-internally spanned it suffices to have two consequent columns with no easy vertices. This happens with probability

$$\left[1-(1-\pi)^{2b}\right]^a \leq \left[1-(1-\pi)^{2a}\right]^a \leq \left[1-(1-\pi)^{2k}\right]^k \leq e^{-\gamma_1/\pi}.$$

To obtain the second inequality we need to look at the function $a \mapsto \left[1-(1-\pi)^{2a}\right]^a$. It decreases to a minimum at some point $\propto \frac{1}{\pi}$, and then increases. We can fix B such that this minimum is at $\frac{10B}{q}$, and the inequality follows. □

We will now bound the number of pre-internally spanned rectangles.

DEFINITION 3.2.18. Fix $l \in \mathbb{N}$. $n_l = n_l(\omega)$ will be the number of pre-internally spanned rectangle inside $[-l, l]^2$ whose longest side is of length between k and $2k+2$.

CLAIM 3.2.19. There exists $l_0 = l_0(\omega)$ such that for all $l \geq l_0$

$$n_l \leq 40k^2\, e^{-\gamma_1/\pi} l^2.$$

PROOF. First, we can bound the number of rectangle of longest side in $[k, 2k+2]$ inside $[-l, l]^2$ whose longest side is of length between k and $2k+2$. Each of them could have one of $(2l+1)^2$ points to be its right corner, at most $2k+2$ options for its height and at most $2k+2$ for its width. That is, the total number of possibilities is at most $20k^2\, l^2$. By FKG, we can take $20k^2\, l^2$ independent rectangles, thus $\nu\left(n_l > 20k^2\, e^{-\gamma_1/\pi} l^2\right)$ is dominated by the probability that a binomial of parameters $20k^2\, l^2$ and $e^{-\gamma_1/\pi}$ is greater than $40k^2\, l^2\, e^{-\gamma_1/\pi}$. By large deviation estimate this happens with probability at most e^{-cl^2} for some c that depends on γ_1, π, k but not on l. We then conclude by the Borel-Cantelli lemma. □

This is the information that we need about the quenched environment ω. We will now consider the state η as well.

LEMMA 3.2.20. *Fix l, $m \le l$. If $[-l,l]^2$ in internally spanned, then it contains an internally spanned rectangle whose longest side is of length between m and $2m+2$.*

PROOF. We will define a sequence $\{\mathcal{C}_n\}_{n=0}^N$ of collections of (possibly overlapping) rectangles in $[-l,l]^2$, such that the following conditions hold

(1) For all n and all rectangles $R \in \mathcal{C}_n$, R is internally spanned or a single easy site.
(2) For all n, the union $\cup_{R\in\mathcal{C}_n} R$ contains all empty sites and all easy sites.
(3) For all n and for all $R, R' \in \mathcal{C}_n$, $R \nsubseteq R'$.
(4) Let l_n be the maximal side length of a rectangle in $\mathcal{C}_n$. Then $l_{n+1} \le 2l_n + 2$.
(5) $\mathcal{C}_N = \left\{[0, L-1]^2\right\}$.

The construction is the same as in **[2]** – start with all empty and easy sites (we assume that $[-l,l]^2$ is internally spanned, so some must exist). Then we make the following observation:

OBSERVATION 3.2.21. Assume that $\mathcal{C}_n$ satisfies conditions 1,2 and 3 above, and that it contains more than one rectangle. Then there are two rectangles in $\mathcal{C}_n$ with distance at most 2.

PROOF. Assume the contrary, so in particular $[-l,l]^2 \setminus \cup_{R\in\mathcal{C}_n} R$ is nonempty. Take the first $x \in [-l,l]^2 \setminus \cup_{R\in\mathcal{C}_n} R$ that is emptied by the bootstrap percolation. This x is difficult, and has has one neighbor in $\cup_{R\in\mathcal{C}_n} R$, which is a contradiction. □

Using this observation, we could, starting at $\mathcal{C}_n$, take two rectangles $R_1, R_2 \in \mathcal{C}_n$ at distance at most 2. Let R be the minimal rectangle containing both of them. Then add R to $\mathcal{C}_n$ and remove all rectangles of $\mathcal{C}_n$ contained in R (among them R_1 and R_2, but possible others as well). This will give us a new collection $\mathcal{C}_{n+1}$ satisfying conditions 1,2 and 3. Since at each step the number of rectangles decreases, and as long as $|\mathcal{C}_n| \ge 2$ we can go on, until we reach some step N at which $|\mathcal{C}_N| = 1$.

Property 4 is easily verified from the construction, so we are left to check property 5. Since $|\mathcal{C}_N| = 1$, we can write $\mathcal{C}_n = \{R\}$ for some rectangle R. We know that if there are sites outside R they must be difficult, but then they could never be emptied, contradicting the fact that $[l]^2$ is internally spanned.

Finally, by property 4, we can take the last n for which $l_n \le m$. Then $m \le l_{n+1} \le 2l_n + 2 \le 2m+2$. □

The connection between τ_0 and internally spanned rectangles is given in the following claim:

CLAIM 3.2.22. Recall the definition of τ_0 (Definition 2.2.1). There must be (for the initial configuration) an internally spanned rectangle containing the origin whose longest side is of length greater or equal τ_0.

PROOF. Consider the span of $[-\tau_0, \tau_0]^2$. It is compound of disjoint rectangles, one of which containing the origin. We know that the origin is emptiable using only the sites of $[-\tau_0, \tau_0]^2$, but not the sites of $[\tau_0 - 1, \tau_0 - 1]^2$. That is, the rectangle containing the origin has a site on the boundary. In particular, its longest edge is of length at least τ_0. □

We can now put everything together and prove equation (3.2.2).

First, we bound the probability that $\tau_0 < k$ –

$$\mu(\tau_0 < k) \leq \mu([-k, k] \text{ contains an empty site}) \leq 4k^2 q = \frac{4B^2}{\pi^2} q.$$

If $\tau_0 \geq k$, by Claim 3.2.5 and Lemma 3.2.20 there must be an internally spanned rectangle of longest side between k and $2k+2$ in $[-\tau_0, \tau_0]^2$. There are two ways for such a rectangle to be internally spanned – either it is pre-internally spanned and contains an empty site, or it is not, and in this case it must contain at least two empty sites. Let $l = \frac{\pi^2 e^{\gamma_1/2\pi}}{2\sqrt{q}} a$. Then, assuming $l \geq l_0$,

$$\begin{aligned}
\mu(\tau_0 \leq l) &\leq \frac{4B^2}{\pi^2} q + n_l\, (2k+2)^2\, q + l^2\, (2k+2)^4\, q^2 \\
&\leq \frac{4B^2}{\pi^2} q + 50 k^4 \pi^4\, a^2 + 5\, k^4 \pi^4 e^{\gamma_1/\pi}\, a^2 q \\
&= \left(\frac{4B^2}{\pi^2} + 5\, B^4 e^{\gamma_1/\pi}\, a^2 \right) q + 50 B^4\, a^2.
\end{aligned}$$

Setting $\gamma = \gamma_1/2$, $a_0 = 2 l_0 \pi^{-2} e^{-\gamma/\pi}$, $C_1 = \left(\frac{4B^2}{\pi^2} + 5\, B^4 e^{\gamma_1/\pi} \right)$ and $C_2 = 50 B^4$ finishes the proof. □

3.3. Fredrickson-Andersen model on $\mathbb{Z}^2$ with threshold 1 or 2

We will now study the KCM on $\mathbb{Z}^2$ defined by the constraints in equations (3.1.1) and (3.1.2). We will see that the hitting time τ_0 scales polynomially with q, like in the bootstrap percolation. However, unlike the bootstrap percolation, this exponent will not be that of the FA1f constraint, but a random exponent changing from one realization of ω to the other.

THEOREM 3.3.1. *Consider the KCM on $\mathbb{Z}^2$ with the constraints defined in equations (3.1.1) and (3.1.2).*

(1) There exists a constant $c > 0$ such that, ν-a.s., the relaxation time of the process is greater than $e^{c/q}$.

(2) ν-a.s. there exist $\underline{\alpha}$ and $\overline{\alpha}$ (with $\underline{\alpha} \leq \overline{\alpha}$) such that

$$\mathbb{P}_\mu \left[\tau_0 \geq q^{-\overline{\alpha}} \right] \xrightarrow{q \to 0} 0, \tag{3.3.1}$$

$$\mathbb{P}_\mu \left[\tau_0 \leq q^{-\underline{\alpha}} \right] \xrightarrow{q \to 0} 0. \tag{3.3.2}$$

Moreover, $\mathbb{E}_\mu[\tau_0] \geq q^{-\underline{\alpha}}$ for q small enough. $\underline{\alpha}$ and $\overline{\alpha}$ depend on ω, and assuming $\pi < e^{-1}$ they could be chosen such that

$$\nu\left(\overline{\alpha} > 11 \frac{\log 1/\pi}{\pi}\, a \right) < e^{-a},$$

$$\nu\left(\underline{\alpha} < \frac{a'}{\pi}\right) < a',$$

for any $a > 3$ and any a' such that $c\pi \leq a' \leq C$ for two constants c, C. In particular, they cannot be deterministic – there exists α_0 such that $\nu\left(\overline{\alpha} < \alpha_0\right) > 0$ but $\nu\left(\underline{\alpha} < \alpha_0\right) < 1$.

This theorem suggests the following conjecture:

CONJECTURE 3.3.2. *ν-almost surely the limit $\lim_{q\to 0} \frac{\log \tau_0}{\log 1/q}$ exists. Its value α is a random variable whose law depends on π. Moreover, the law of $\pi\alpha$ converges (in some sense) to a non-trivial law as π tends to 0.*

REMARK 3.3.3. Like in the case of the bootstrap percolation, we can see how the time scales fit with the deterministic models when taking π to 1 or to 0. For $\pi = 1$ we obtain the FA1f, in which τ_0 is polynomial, of order q^{-2} [1]. In the other limit, for small π, the scale obtained for τ_0 is (up to logarithmic corrections) $q^{1/\pi}$. If we believe that π could be taken of order q (though, strictly speaking, the theorem does not allow us to do that), we obtain (up to logarithmic corrections) the scaling $e^{1/q}$, just like in FA2f [**43**].

3.3.1. Spectral gap (proof of part 1 of Theorem 3.3.1). The spectral gap of this model is dominated by that of the FA2f model. Fix any γ strictly greater than the gap of FA2f. Then there is a local non-constant function f such that

$$\frac{\mathcal{D}^{\text{FA2f}} f}{\text{Var} f} \leq \gamma,$$

where $\mathcal{D}^{\text{FA2f}}$ is the Dirichlet form of the FA2f model.

Since f is local, it is supported in some square of size $L \times L$, for L big enough. ν-almost surely it is possible to find a far away square in $\mathbb{Z}^2$ of size $L \times L$ that contains only difficult sites. By translation invariance of the FA2f model we can assume that this is the square in which f is supported. In this case, $\mathcal{D}f = \mathcal{D}^{\text{FA2f}} f$, and this shows that indeed the gap of the model with random threshold is smaller than that of FA2f, which by [**19**] is bounded by $e^{-c/q}$. □

3.3.2. Upper bound (proof of equation (3.3.1)). For the proof of the upper bound we will use Corollary 2.2.2 in order to bound τ_0 by a path argument. As in the proof of the upper bound for the bootstrap percolation, we will consider the good squares (see Definition 3.2.4) and their infinite cluster. In fact, by Claim 3.2.5, by choosing L big enough we may assume that the box $[L]^2$ is in this cluster:

$$L = \inf\left\{d \,:\, [d]^2 \text{ is in an infinite good cluster}\right\}. \tag{3.3.3}$$

Let us fix this L until the end of this proof.

[1] It is shown in [**19**] that the spectral gap of FA1f in $\mathbb{Z}^2$ scales as q^{-2} up to logarithmic corrections. Using their methods together with those presented in the proof of Theorem 3.3.1 one finds the same scaling also for τ_0.

CLAIM 3.3.4. Assume that $\pi < e^{-1}$. Then $\nu\left(L > 2\frac{\log 1/\pi}{\pi}a\right) < e^{-a}$ for all $a > 3$.

PROOF. Fix $d > 2\frac{\log 1/\pi}{\pi}a$. Then, by Claim 3.2.5, the probability that a box of size d is good is greater than $1 - 20e^{-2a}$. Then (see, e.g., [**22, 32**]), the probability that $[d]^2$ does not belong to the infinite cluster is at most $320e^{-2a}$, and the result follows. □

We will also choose an infinite self avoiding path of good boxes starting at the origin and denote it by $i_0, i_1, i_2, \ldots$. Note that this path depends on ω but not on η.

On this cluster empty sites will be able to propagate, and the next definition will describe the seed needed in order to start this propagation.

DEFINITION 3.3.5. A box in $\mathbb{Z}^2$ is *essentially empty* if it is good and contains an entire line or an entire column of empty sites. This will depend on both ω and η.

In order to guarantee the presence of an essentially empty box we will fix

$$l = q^{-L-1}, \tag{3.3.4}$$

and define the bad event

DEFINITION 3.3.6. $B = \{\text{none of the boxes } i_0, \ldots, i_l \text{ is essentially empty}\}$. For fixed ω the path $i_0, i_1, i_2, \ldots$ is fixed, and B is an event in Ω.

A simple bound shows that

$$\mu(B) \leq \left(1 - q^L\right)^l \leq e^{-1/q}. \tag{3.3.5}$$

We can use this inequality in order to bound the hitting time at B:

CLAIM 3.3.7. There exists $C > 0$ such that $\mathbb{P}_\mu(\tau_B \leq t) \leq Ce^{-1/q}t$.

PROOF. We use the graphical construction of the Markov process. In order to hit B, we must hit it at a certain clock ring taking place in one of the sites of $\cup_{n=1}^{l}\left(Li + [L]^2\right)$. Therefore,

$$\begin{aligned}\mathbb{P}(\tau_B \leq t) &\leq \mathbb{P}\left[\text{more than } 2(2L+1)^2 lt \text{ rings by time } t\right] + 2(2L+1)^2 l\, t\, \mu(B)\\ &\leq e^{-(2L+1)^2 q^{-L-1} t} + 2(2L+1)^2 q^{-L-1}\, t\, e^{-1/q} \leq Ce^{-1/q}t.\end{aligned}$$

□

In order to bound τ_0 we will study the hitting time of

$$A = B \cup \{\eta(0) = 0\}. \tag{3.3.6}$$

LEMMA 3.3.8. *Fix $\eta \in \Omega$. Then there exists a path $\eta_0, \ldots, \eta_N$ of configurations and a sequence of sites $x_0, \ldots x_{N-1}$ such that*

(1) $\eta_0 = \eta$,
(2) $\eta_N \in A$,
(3) $\eta_{i+1} = \eta_i^{x_i}$,

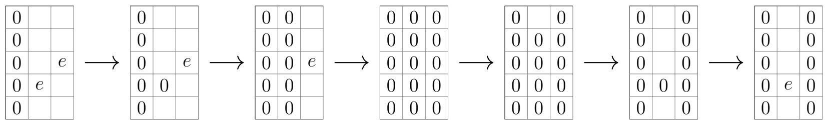

FIGURE 3.3.1. Illustration of the proof of Lemma 3.3.8. We see here how to create an empty column and propagate it using the easy sites. 0 represents an empty site, otherwise the state is the initial one. e stands for an easy site.

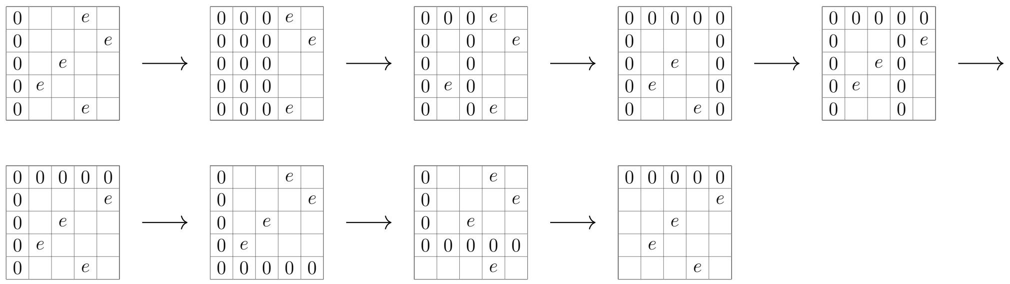

FIGURE 3.3.2. Illustration of the proof of Lemma 3.3.8. We see here how to rotate an empty column in a good box. 0 represents an empty site, otherwise the state is the initial one. e stands for an easy site.

(4) $c_{x_i}(\eta_i) = 1$,
(5) $N \leq 4L^2 l$,
(6) For all $i \leq N$, η_i *differs from* η *at at most* $3L$ *points, contained in at most two neighboring boxes.*

PROOF. If $\eta \in A$, we take the path η with $N = 0$. Otherwise $\eta \in B^c$, so there is an essentially empty box in $i_0, \ldots, i_l$, which then contains an empty column (or row). We can then create an empty column (row) next to it and propagate that column (row) as in figure 3.3.1. When the path rotates we can rotate this propagating column (row) as show in figure 3.3.2. □

We can use this path together with Corollary 2.2.2 in order to bound τ_A.

LEMMA 3.3.9. *There exists* $C_L > 0$ *(that may depends on* L *but not on* q*) such that* $\mu(\tau_A) \leq C_L\, L^{10L}\, q^{-5L-2}$.

PROOF. Since τ vanishes on A, taking the path defined in Lemma 3.3.8,

$$\tau_A(\eta) = \sum_{i=0}^{N-1} \left(\tau_A(\eta_i) - \tau_A(\eta_{i+1})\right).$$

In the following we use the notation

$$\nabla_x \tau_A(\eta) = \tau_A(\eta) - \tau_A(\eta^x).$$

Then, by Cauchy Schwartz inequality,

$$\begin{aligned}\mu(\tau_A)^2 &\leq \mu(\tau_A^2) = \sum_\eta \mu(\eta)\left(\sum_{i=0}^{N-1} \nabla_{x_i}\tau_A(\eta_i)\right)^2 \\ &\leq \sum_\eta \mu(\eta) N \sum_i c_{x_i}(\eta_i)\left(\nabla_{x_i}\tau_A(\eta_i)\right)^2 \\ &= \sum_\eta \mu(\eta) N \sum_i \sum_z \sum_{\eta'} c_z(\eta')\left(\nabla_z\tau_A(\eta')\right)^2 \mathbb{1}_{z=x_i}\mathbb{1}_{\eta'=\eta_i}.\end{aligned}$$

By property number 6 of the path, we know that $\mu(\eta) \leq q^{-3L}\mu(\eta')$, yielding

$$\mu(\tau_A)^2 \leq q^{-3L} N \sum_{\eta'} \mu(\eta') \sum_z c_z(\eta')\left(\nabla_z\tau_A(\eta')\right)^2 \sum_i \mathbb{1}_{z=x_i} \sum_\eta \mathbb{1}_{\eta'=\eta_i}.$$

Still using property 6, η differs form η' at at most $3L$ point, all of them in the box containing z or in one of the two neighboring boxes. This gives the bound $\sum_\eta \mathbb{1}_{\eta'=\eta^{(i)}} \leq (3L^2)^{3L}$. Finally, bounding $\mathbb{1}_{z=x_i}$ by 1,

$$\begin{aligned}\mu(\tau_A)^2 &\leq q^{-3L}\left(3L^2\right)^{3L} N^2 \sum_{\eta'} \mu(\eta') \sum_z c_z(\eta')\left(\nabla_z\tau_A(\eta')\right)^2 \\ &\leq 16\left(3L^2\right)^{3L} L^4 q^{-5L-2}\,\mathcal{D}\tau_A.\end{aligned}$$

This conclude the proof of the lemma by Corollary 2.2.2. □

Using this lemma and the bound on τ_B in Claim 3.3.7, we can finish the estimation of the upper bound. By Markov inequality and the fact that $\mu(\tau_A) = \mathbb{E}_\mu(\tau_A)$ (recall equation (2.2.2))

$$\mathbb{P}_\mu\left(\tau_A \geq C_L\, q^{-5L-3}\right) \leq q.$$

On the other hand, by Claim 3.3.7,

$$\begin{aligned}\mathbb{P}_\mu\left(\tau_A < C_L\, q^{-5L-5}\right) &\leq \mathbb{P}_\mu\left(\tau_0 < C_L\, q^{-5L-3}\right) + \mathbb{P}_\mu\left(\tau_B < C_L\, q^{-5L-3}\right) \\ &\leq \mathbb{P}_\mu\left(\tau_0 < C_L\, q^{-5L-3}\right) + C_L'\, e^{-1/q}.\end{aligned}$$

Therefore

$$\mathbb{P}_\mu\left(\tau_0 \geq C_L\, q^{-5L-3}\right) \leq q + C_L e^{-1/q},$$

and taking $\overline{\alpha} = 5L+3$ will suffice. □

3.3.3. Lower bound (proof of equation (3.3.2)**).** In order to prove equation (3.3.2) we will show that before hitting $\{\eta_0 = 0\}$ we must pass through a very unlikely state, in which many sites are empty.

Definition 3.3.10. Consider a rectangle $R = [x_1, y_1] \times [x_2, y_2]$. We say that R is *d-difficult* if it contains at least d distinct pairs of neighboring lines or d distinct pairs of neighboring columns with only difficult vertices. Being d-difficult depends only on the disorder ω, and not on η.

CLAIM 3.3.11. If R is d-difficult and internally spanned, than it must contain at least d empty sites.

PROOF. We may assume without loss of generality that R contains d distinct pairs of neighboring lines with only difficult vertices. That is, there exist $i_1 < i_2 < \cdots < i_d$ such that $i_{n+1} > i_n + 1$ and the sites of $R \cap (\mathbb{Z} \times \{i_n, i_n + 1\})$ are difficult for all $n \le d$. We will show that $R \cap (\mathbb{Z} \times \{i_n, i_n + 1\})$ contains an empty site for all $n \le d$, and this will prove the claim.

Fix $n \le d$, and assume that the sites of $R \cap (\mathbb{Z} \times \{i_n, i_n + 1\})$ are all occupied. Since R is internally spanned, one of these sites must be emptied at some time. However, at this time it could only have one empty neighbor in R – three of its neighbors are in $(\mathbb{Z} \times \{i_n, i_n + 1\})$ and the sites in $R \cap (\mathbb{Z} \times \{i_n, i_n + 1\})$ are occupied by minimality. This is a contradiction, since this site must be difficult. □

In order to guarantee the existence of d-difficult rectangles, we define the event $D_{L,d}$:

DEFINITION 3.3.12. Let $L, d \in \mathbb{N}$. The event $D_{L,d}$ is the intersection of the following events:

(1) $\#\left\{0 \le i \le \left\lfloor \frac{L}{2} \right\rfloor \text{ : all vertices in } \{2i, 2i+1\} \times [-L, L] \text{ are difficult}\right\} \ge d$,
(2) $\#\left\{0 \le i \le \left\lfloor \frac{L}{2} \right\rfloor \text{ : all vertices in } \{-2i, -2i-1\} \times [-L, L] \text{ are difficult}\right\} \ge d$,
(3) $\#\left\{0 \le i \le \left\lfloor \frac{L}{2} \right\rfloor \text{ : all vertices in } [-L, L] \times \{2i, 2i+1\} \text{ are difficult}\right\} \ge d$,
(4) $\#\left\{0 \le i \le \left\lfloor \frac{L}{2} \right\rfloor \text{ : all vertices in } [-L, L] \times \{-2i, 2i-1\} \text{ are difficult}\right\} \ge d$.

Note that this event depends only on the disorder ω.

OBSERVATION 3.3.13. Under $D_{L,d}$, all rectangles in $[-L, L]^2$ that contain the origin and at least one site on the boundary are d-difficult.

CLAIM 3.3.14. There exist two positive constants c, C such that for all a satisfying $c\pi \le a \le C$,

$$\nu\left(\exists d \ge \frac{a}{\pi} \text{ and } L \ge \frac{10a}{\pi} \text{ such that } D_{L,d}\right) \ge 1 - a.$$

PROOF. Let $0 < b < 2\pi$, fix $L = \left\lfloor \frac{b}{4\pi} - \frac{1}{2} \right\rfloor$ and $d = \left\lfloor \frac{L}{4} e^{-b} \right\rfloor$. We will estimate the probability of $D_{L,d}$. For fixed i, the probability that $\{2i, 2i+1\} \times [-L, L]$ contain only difficult sites equals $(1-\pi)^{2L+1} \ge e^{-2(2L+1)\pi} \ge e^{-b}$. Therefore, the first event in the definition of $D_{L,d}$ (3.3.12) is described by a binomial random variable with parameters $\left(\left\lfloor \frac{L}{2} \right\rfloor, e^{-b}\right)$ that must be greater than d. This probability is bounded from below by $1 - e^{-\frac{1}{4}e^{-b}L}$. Then, by the union bound,

$$\nu\left(D_{L,d}\right) \ge 1 - 2Le^{-\frac{1}{4}e^{-b}L}.$$

We may choose b (uniformly in π) such that $a = \pi d$, and if it is small enough then $2Le^{-\frac{1}{4}e^{-b}L} \le a$. □

CLAIM 3.3.15. Fix L, d and $x \in \mathbb{Z}^2$. Let $G_x \subseteq \Omega$ be the event that the origin is in the span of $[-L, L]^2$ for η, but it is not in the span of $[-L, L]^2$ for η^x (recall Definition 1.3.2). Let

$G = \cup_{x\in\mathbb{Z}^2} G_x$. Assume further that $D_{L,d}$ occurs. Then

$$\mu\left(G\right) \leq \binom{2L^2+1}{d} q^d.$$

PROOF. First, note that x must be on the boundary of $[-L,L]^2$, and that it is contained in the same internally spanned rectangle as the origin for the configuration η. Then by Claim 3.3.11 and Observation 3.3.13 the configuration η contains at least d empty sites. □

We can now conclude the proof for the lower bound. Fix d, L and assume that $D_{L,d}$ occurs. The argument of Claim 3.3.7 tells us that τ_G (for the event G defined in Claim 3.3.15) is greater than q^{-d+1} with probability that tends to 1 as $q \to 0$ (while keeping ω, and therefore L, fixed). If we start with a configuration for which the origin is not in the span of $[-L,L]^2$, it could only be emptied after τ_G – at the first instant in which the span of $[-L,L]^2$ includes the origin, G_x must occur for the site that has just been flipped. Since the probability to start with an entirely occupied $[-L,L]^2$ tends to 1 as $q \to 0$, equation (3.3.2) is satisfied for $\underline{\alpha} = d-1$.

In order to bound also the expected value of τ_0 we will use Proposition 2.2.8. Let us consider the function

$$f = \mathbb{1}_{0\text{ is not in the span of }[-L,L]^2}.$$

We can bound its Dirichlet form using Claim 3.3.15:

$$\begin{aligned}\mathcal{D}f &= \mu\left(\sum_x c_x \mathrm{Var}_x f\right) \leq \mu\left(\sum_x c_x\, q\mathbb{1}_{G_x}\right)\\ &\leq q\, 4\left(2L+1\right)\binom{2L^2+1}{d} q^d = C_L q^{d+1}.\end{aligned}$$

The expected value is bounded by the probability that all sites are occupied –

$$\mu f \geq \left(1-q\right)^{(2L+1)^2}.$$

Now consider for some $\lambda \in \mathbb{R}$ the rescaled function $\overline{f} = \lambda f$. Recall Definition 2.2.7.

$$\begin{aligned}\mathcal{T}\overline{f} &= 2\mu\overline{f} - \mathcal{D}\overline{f}\\ &\geq 2\lambda\left(1-q\right)^{(2L+1)^2} - \lambda^2\, C_L\, q^{d+1}.\end{aligned}$$

The optimal choice of λ is $\frac{(1-q)^{(2L+1)^2}}{C_L\, q^{d+1}}$, which yields

$$\mathcal{T}\overline{f} \geq \frac{\left(1-q\right)^{2(2L+1)^2}}{C_L} q^{-d-1}.$$

In addition, $\overline{f} \in V_{\{\eta_0=0\}}$, so by Proposition 2.2.8

$$\mu\left(\tau_0\right) \geq \frac{\left(1-q\right)^{2(2L+1)^2}}{C_L} q^{-d-1}.$$

□

3.4. Fredrickson-Andersen model on $\mathbb{Z}^d$ with threshold 1 or d

The results of the previous section could be extended to $\mathbb{Z}^d$. In [**47**] it is explained how the proof in section 3.3 could be generalized, but here we will use a different strategy. We will see that τ_0 scales as a power law (like in the FA1f case [**19**]), with a power that may change from one realization to the other. We will not analyze the scaling of this power when π is small as we did for the two dimensional case, but we can expect an iteretated exponent scaling that would fit the FAdf model when π is of order q (see Remark 3.3.3).

THEOREM 3.4.1. *Consider the KCM on $\mathbb{Z}^d$ with the constraints defined in equations (3.1.1) and (3.1.2).*

1. *There exists a constant $c > 0$ such that, ν-a.s., the relaxation time of the process is greater than $\exp_{(d-1)}\left(\frac{c}{q}\right)$, where $\exp_{(\cdot)}$ is the iterated exponential.*
2. *Recall Definition 2.2.4 and let $\overline{\tau} = \overline{\tau}_{\{\eta_0=0\}}$. ν-a.s. there exist $\underline{\alpha}$ and $\overline{\alpha}$ (with $\underline{\alpha} \leq \overline{\alpha}$) such that for q small enough*
$$\overline{\tau} \leq q^{-\overline{\alpha}} \tag{3.4.1}$$
and
$$\mathbb{E}_{\mu}\left[\tau_0\right] \geq q^{-\underline{\alpha}}. \tag{3.4.2}$$

REMARK 3.4.2. As already mentioned in the definition of the model, a more general disorder would allow any threshold between 1 and d, and not just d. It will make notations more complicated, but by monotonicity the theorem will also hold for that case.

3.4.1. Spectral gap (proof of part 1 of Theorem 3.4.1). Just like in the two dimensional case, the gap is bounded by that of the FAdf model on $\mathbb{Z}^d$, which by [**43**] yields the result. □

3.4.2. Upper bound (proof of equation (3.4.1)). In order to prove equation (3.4.1) we will use Claim 2.2.12 together with the following proposition proven in [**40**].

PROPOSITION 3.4.3. *Consider a KCM on $\mathbb{Z}$ with spin values in $\mathcal{S}$ and equilibrium measure μ, which is a product of copies of the measure μ_0 on $\mathcal{S}$. Let G be an event on $\mathcal{S}$, and consider the generator defined on local functions as*
$$\mathcal{L}^G f = \sum_{i\in\mathbb{Z}} \mathbb{1}_{G_{i-1}\cup G_{i+1}}\left(\mu_i f - f\right).$$
Then the gap of $\mathcal{L}^G$ is greater than $C\,\mu_0\left(G\right)^3$ for a constant $C > 0$.

We start by extending Definition 3.2.4 to dimension d:

DEFINITION 3.4.4. A square (that is, a subset of $\mathbb{Z}^2$ of the form $x + [L]^d$) is *good* if each of its $d-1$ dimensional sections could be emptied with an initial configuration in which one of its two neighboring faces is empty and all other vertices are occupied. Note that being good depends only on the disorder ω and not on the state η.

From bootstrap percolation results (e.g., **[7]**) we know that

CLAIM 3.4.5. The probability that a square of size $[L]^d$ is good tends to 1 as L tends to ∞.

This implies that ν-a.s. we can choose L (depending on ω) such that the origin belong to an infinite cluster of good squares. We fix this L, and a biinfinite nearest neighbor self avoiding path $\dots, b_{-1}, b_0, b_1, \dots \in \mathbb{Z}^d$, such that B_{b_i} is good for all $i \in \mathbb{Z}$. In the following we will consider the dynamics restricted to the graph $\cup_{i\in\mathbb{Z}} B_{b_i}$ (see Definition 2.2.10), and denote B_{b_i} by B_i.

First, we define the coarse grained dynamics. Let E_i be the event, that for each of the d directions, at least one section of the box B_i is empty. Then we define

$$\begin{aligned}
c_i^{\mathrm{CG}} &= \mathbb{1}_{E_{i+1}\cup E_{i-1}},\\
\mathcal{L}^{\mathrm{CG}} f &= \sum_i c_i^{\mathrm{CG}} \left(\mu_{B_i} f - f\right),\\
\mathcal{D}^{\mathrm{CG}} f &= \mu\left(\sum_i c_i^{\mathrm{CG}} \mathrm{Var}_{B_i} f\right).
\end{aligned}$$

OBSERVATION 3.4.6. If we take $\mathcal{S} = \{0,1\}^{B_0}$ with the measure μ_{B_0}, and $G = E_0$, then $\mathcal{L}^G$ given in Proposition 3.4.3 describes the same dynamics as $\mathcal{L}^{\mathrm{CG}}$. In particular,

$$\operatorname{gap}\left(\mathcal{L}^{\mathrm{CG}}\right) \geq \mu\left(E_0\right)^3 \geq C\, q^{3dL^{d-1}}.$$

The next step is to bound $\mathcal{D}$ using $\mathcal{D}^{\mathrm{CG}}$. First, we write

$$c_i^{\mathrm{CG}} \leq \mathbb{1}_{E_{i+1}} + \mathbb{1}_{E_{i-1}}.$$

Consider the term $\mathbb{1}_{E_{i+1}}$. Without loss of generality assume $b_{i+1} = b_i + e_1$. Let $S_k = Lb_i + \{k\} \times [L]^{d-1}$ be the $k$the section of B_i in the direction e_1. By the definition of E_{i+1},

$$\mathbb{1}_{E_{i+1}} \leq \sum_{n=L+1}^{2L} \mathbb{1}_{S_n \text{ is empty}}. \tag{3.4.3}$$

In order to bound the term $\mathbb{1}_{S_n \text{ is empty}} \mathrm{Var}_{B_i} f$ we will use the a dynamics in $Lb_i + [n-1] \times [L]^{d-1}$, with empty boundary on $Lb_i + \{n\} \times [L]^{d-1}$ and occupied elsewhere.

$$\begin{aligned}
c_k^{\mathrm{S}} &= \mathbb{1}_{\{S_{k+1} \text{ is empty}\}\cup\{S_{k-1} \text{ is empty}\}},\\
\mathcal{L}^{\mathrm{S}n} f &= \sum_{k=1}^{n-1} c_k^{\mathrm{S}} \left(\mu_{S_k} f - f\right),\\
\mathcal{D}^{\mathrm{S}n} f &= \mu_{\cup_{k=1}^{n-1} S_k}\left(\sum_{k=1}^{n-1} c_k^{\mathrm{S}} \mathrm{Var}_{S_k} f\right).
\end{aligned}$$

OBSERVATION 3.4.7. If we take $\mathcal{S} = \{0,1\}^{S_0}$ with the measure μ_{S_0}, and $G = \{S_0 \text{ is empty}\}$, then $\mathcal{L}^G$ given in Proposition 3.4.3 describes the same dynamics as $\mathcal{L}^{\mathrm{S}n}$ on $Lb_i + [n-1] \times [L]^{d-1}$. In particular, due to the empty boundary,

$$\operatorname{gap}\left(\mathcal{L}^{\mathrm{S}n}\right) \geq C\, \mu\left(S_0 \text{ is empty}\right)^3 = C\, q^{3L^{d-1}}.$$

Before continuing, we mention the following general property of the spectral gap.

CLAIM 3.4.8. Take any Markov process on a finite state space Ω, reversible with respect to a measure μ. Assume that it is ergodic, and that for all transitions (η, η') with non-zero rate $R(\eta, \eta') \geq r$, for some fixed $r > 0$ (recall equations 2.1.1 and 2.1.3). Then the spectral gap is at least $\frac{r}{2|\Omega|^2}$.

PROOF. Let $f : \Omega \to \mathbb{R}$ such that $\mu(f) = 0$ and $\mu(f^2) = 1$. There exists $\eta_+ \in \Omega$ such that $|f(\eta_+)| \geq 1$. Assume without loss of generality $f(\eta_+) > 0$. $\mu(f) = 0$, so there must be $\eta_- \in \Omega$ such that $f(\eta') < 0$. By ergodicity, there is a path $\eta_+ = \eta_0, \ldots, \eta_N = \eta_-$ with non-zero rate transitions, and length $N \leq |\Omega|$. $f(\eta_N) - f(\eta_0) > 1$, so there must be two configurations η_i, η_{i+1} such that $f(\eta_{i+1}) - f(\eta_i) > \frac{1}{|\Omega|}$. Therefore

$$\mathcal{D}f = \frac{1}{2}\sum_{\eta,\eta'} R(\eta, \eta')\left(f(\eta) - f(\eta')\right)^2 \geq \frac{1}{2} r\, \left(f(\eta_i) - f(\eta_{i+1})\right)^2 \geq \frac{r}{2|\Omega|^2}.$$

□

We will use Claim 3.4.8 on the dynamics restricted to S_k, with the Dirichlet form given by

$$\mathcal{D}_{S_k} f = \mu_{S_k}\left(\sum_{x\in S_k} c_x \mathrm{Var}_x f\right).$$

We will denote by $\overline{\mathcal{D}}_k$ the Dirichlet form of the same dynamics, but with empty boundary at S_{k+1} and occupied elsewhere. Note that this is the exact same dynamics as the one we obtain by taking F_{k-1} empty and occupied elsewhere, and that this dynamics is ergodic since all boxes are good.

With these definitions, taking $r = q^{L^{d-1}}$ in Claim 3.4.8, the gap of the $\overline{\mathcal{D}}_k$-dynamics is at least $2^{L^{d-1}-1} q^{L^{d-1}}$. Therefore,

$$c_k^{\mathrm{S}} \mathrm{Var}_{S_k} f \leq 2^{-L^{d-1}+1} q^{-L^{d-1}}\, c_k^{\mathrm{S}} \mathcal{D}_{S_k} f.$$

We now use Observation 3.4.7, equation (3.4.3) and the above equation to obtain

$$\begin{aligned}
\mu\left(\mathbb{1}_{E_{i+1}} \mathrm{Var}_{B_i} f\right) &\leq \sum_{n=L+1}^{2L} \mu\left(\mathbb{1}_{S_n \text{ is empty}} \mathrm{Var}_{B_i} f\right) \\
&\leq \sum_{n=L+1}^{2L} \mu\left(\mathbb{1}_{S_n \text{ is empty}} \mathrm{Var}_{\cup_{k=1}^{n-1} S_k} f\right) \\
&\leq C^{-1} q^{-3L^{d-1}} \sum_{n=L+1}^{2L} \mu\left(\mathbb{1}_{S_n \text{ is empty}} \mathcal{D}^{\mathrm{S}n} f\right) \\
&= C^{-1} q^{-3L^{d-1}} \sum_{n=L+1}^{2L} \mu\left(\mathbb{1}_{S_n \text{ is empty}} \sum_{k=1}^{n-1} c_k^{\mathrm{S}} \mathrm{Var}_{S_k} f\right)
\end{aligned}$$

$$\begin{aligned}
&\leq C^{-1}q^{-3L^{d-1}}2^{-L^{d-1}+1}q^{-L^{d-1}}\sum_{n=L+1}^{2L}\mu\left(\mathbb{1}_{S_n\text{ is empty}}\sum_{k=1}^{n-1}c_k^{\mathrm{S}}\mathcal{D}_{S_k}f\right)\\
&=C^{-1}q^{-3L^{d-1}}2^{-L^{d-1}+1}q^{-L^{d-1}}q^{2L^{d-1}}\sum_{n=L+1}^{2L}\sum_{k=1}^{n-1}\mu\left(\mathcal{D}_{S_k}f\right)\\
&=C^{-1}2^{-L^{d-1}+1}q^{-2L^{d-1}}\sum_{n=L+1}^{2L}\sum_{k=1}^{n-1}\mu\left(\sum_{x\in S_k}c_x\mathrm{Var}_x f\right)\\
&\leq C^{-1}2^{-L^{d-1}+1}q^{-2L^{d-1}}L\,\mu\left(\sum_{x\in B_i\cup B_{i+1}}c_x\mathrm{Var}_x f\right).
\end{aligned}$$

By symmetry this bound also holds for $\mu\left(\mathbb{1}_{E_{i-1}}\mathrm{Var}_{B_i}f\right)$.

Finally, using Theorem 3.4.6

$$\begin{aligned}
\mathrm{Var}f &\leq C^{-1}q^{-3dL^{d-1}}\mu\left(\sum_i c_i^{\mathrm{CG}}\mathrm{Var}_{B_i}f\right)\\
&\leq C^{-1}q^{-3dL^{d-1}}\mu\left(\sum_i \mathbb{1}_{E_{i+1}}\mathrm{Var}_{B_i}f\right)+C^{-1}q^{-3dL^{d-1}}\mu\left(\sum_i \mathbb{1}_{E_{i-1}}\mathrm{Var}_{B_i}f\right)\\
&\leq 2C^{-1}\,q^{-3dL^{d-1}}\sum_i 2^{-L^{d-1}+1}q^{-2L^{d-1}}L\sum_{x\in B_i\cup B_{i+1}}c_x\mathrm{Var}_x f\\
&\leq C^{-1}L\,2^{-L^{d-1}+3}\,q^{-(3d+2)L^{d-1}}\sum_{x\in\cup_{i\in\mathbb{Z}}B_i}c_x\mathrm{Var}_x f.
\end{aligned}$$

That is, the spectral gap of the restricted dynamics is greater than $CL^{-1}2^{L^{d-1}+3}\,q^{(3d+2)L^{d-1}}$. Choosing $\overline{\alpha}>(3d+2)\,L^{d-1}$ finishes the proof by Claim 2.2.12. □

3.4.3. Lower bound (proof of equation (3.4.2)**).** The lower bound of this model can by analyzed in a similar way to the case of $\mathbb{Z}^2$. However, if we are not interested in the way $\underline{\alpha}$ scales with π, we can simply observe that the time at which the origin is empty is certainly longer than the that of the unconstrained dynamics. Therefore, any $\underline{\alpha}<1$ will suffice. □

3.5. Mixed North-East and Fredrickson-Andersen 1 spin facilitated models on $\mathbb{Z}^2$

Unlike the models that have been studied in the previous sections, this KCM is not necessarily ergodic. For a fixed environment ω there exists a critical value q_c above which all sites are emptiable for the bootstrap percolation, and below which some sites remain occupied forever.

Denote by p^{SP} the critical probability for the Bernoulli site percolation on $\mathbb{Z}^2$ and by p^{OP} the critical probability for the oriented Bernoulli percolation on $\mathbb{Z}^2$. Then

$$q_c\leq 1-\frac{1-p^{\mathrm{SP}}}{1-\pi},$$

since if we have an infinite cluster of sites that are either easy or empty all sites are emptiable. In particular, if $\pi > p^{\mathrm{SP}}$ the critical probability is 0. On the other hand, if there is an infinite up-right path of difficult sites that are all occupied, this path could never be emptied. This will imply that $q_c \geq 1 - \frac{p^{\mathrm{OP}}}{1-\pi}$.

We will see for this model that it is possible to have an infinite relaxation time, and still the tail of the distribution of τ_0 decays exponentially, with a rate that scales polynomially with q.

THEOREM 3.5.1. *Consider the kinetically constrained model described above, with $\pi > p^{SP}$ and $q < p^{OP}$.*

(1) ν-almost surely the spectral gap is 0, i.e., the relaxation time is infinite.
(2) There exist two positive constants c, C depending on π and a ν-random variable τ such that
  (a) $\mathbb{P}_\mu(\tau_0 \geq t) \leq e^{-t/\tau}$ for all $t > 0$,
  (b) $\nu(\tau \geq t) \leq C\, t^{\frac{c}{\log q}}$ for t large enough.

3.5.1. Spectral gap (proof of part 1 of Theorem 3.5.1). Just like in the proof of Theorem 3.3.1, one can always find arbitrarily large regions of difficult sites, so the gap is bounded by that of the north-east model. Since for the parameters that we have chosen the north-east model is not ergodic, it has 0 gap [**19**]. □

3.5.2. Hitting time (proof of part 2 of Theorem 3.5.1). Let A be the event $\{\eta_0 = 0\}$. Recall Definition 2.2.4 and let

$$\tau = \overline{\tau}_A.$$

The exponential tail of τ_0 is a consequence of Proposition 2.2.5, so we are left with proving that $\nu(\tau \geq t) \leq t^{\frac{c}{\log q}}$ for some constant c. We will do that by choosing a subgraph on which we can estimate the gap, and then apply Claim 2.2.12.

Since π is greater than the critical probability for the Bernoulli site percolation, there will be an infinite cluster of easy sites $\mathcal{C}$. We denote by $\mathcal{C}_0$ the cluster of the origin surrounded by a path in $\mathcal{C}$. $\partial\mathcal{C}_0$ will be the outer boundary of $\mathcal{C}_0$, i.e., the sites in $\mathcal{C}$ that have a neighbor in $\mathcal{C}_0$. Then, we fix a self avoiding infinite path of easy sites $v_0, v_1, \dots$ starting with the sites of $\partial\mathcal{C}_0$. That is, $v_0, \dots, v_{|\partial\mathcal{C}_0|}$ is a path that encircles $\mathcal{C}_0$, and then $v_{|\partial\mathcal{C}_0|+1}, \dots$ continues to infinity. We will denote $\mathcal{V} = \{v_i\}_{i\in\mathbb{N}}$. Let $H = \mathcal{V} \cup \mathcal{C}_0$, and consider the restricted dynamics $\mathcal{L}_H$ introduced in Definition 2.2.10. We split the dynamics in two – for some local function f on H

$$\mathcal{L}_H f = \mathcal{L}^{\mathcal{C}_0} f + \mathcal{L}^{\mathcal{V}} f,$$

$$\mathcal{L}^{\mathcal{V}} = \sum_{i\in\mathbb{N}} c^H_{v_i}\left(\mu_{v_i} f - f\right),$$

$$\mathcal{L}^{\mathcal{C}_0} = \sum_{x\in\mathcal{C}_0} c^H_x\left(\mu_x f - f\right).$$

Note that the boundary conditions of the $\mathcal{C}_0$ dynamics depend on the state of the vertices in $\mathcal{V}$ and vice versa. We will denote by $\mathcal{L}^{\mathcal{C}_0}_0$ the $\mathcal{C}_0$ dynamics with empty boundary conditions and by $\mathcal{L}_1^{\mathcal{V}}$ the $\mathcal{V}$ dynamics with occupied boundary conditions. All generators come with their Dirichlet forms carrying the same superscript and subscript.

We will bound the gap of $\mathcal{L}_H$ using the gaps of $\mathcal{L}_1^{\mathcal{V}}$, $\mathcal{L}_0^{\mathcal{C}_0}$ and the following block dynamics:

$$\mathcal{L}^b f = (\mu_{\mathcal{V}}(f) - f) + \mathbb{1}_{\partial\mathcal{C}_0 \text{ is empty}} (\mu_{\mathcal{C}} f - f) .$$

Denote the spectral gaps of $\mathcal{L}_1^{\mathcal{V}}$, $\mathcal{L}_0^{\mathcal{C}_0}$,$\mathcal{L}^b$, $\mathcal{L}_H$ by $\gamma_1^{\mathcal{V}}$,$\gamma_0^{\mathcal{C}_0}$,γ^b,γ_H.

By Proposition 4.4 of [**19**]:

Claim 3.5.2.

$$\gamma^b = 1 - \sqrt{1 - q^{|\partial\mathcal{C}_0|}},$$

i.e., $\mathrm{Var} f \leq \frac{1}{1-\sqrt{1-q^{|\partial\mathcal{C}_0|}}}\mathcal{D}^b f$ for any local function f.

Let us now use this gap in order to relate γ_H to $\gamma^{\mathcal{V}}$ and $\gamma^{\mathcal{C}_0}$:

Claim 3.5.3.

$$\gamma_H \geq \gamma^b \min\left\{\gamma_1^{\mathcal{V}}, \gamma_0^{\mathcal{C}_0}\right\} .$$

Proof. Fix a non-constant local function f.

$$\begin{aligned}
\mathrm{Var} f &\leq \frac{1}{\gamma^b}\mathcal{D}^b f = \frac{1}{\gamma^b}\left[\mu\left(\mathrm{Var}_{\mathcal{V}} f\right) + \mu\left(\mathbb{1}_{\partial\mathcal{C}_0 \text{ is empty}} \mathrm{Var}_{\mathcal{C}} f\right)\right] \\
&\leq \frac{1}{\gamma^b}\left[\frac{1}{\gamma_1^{\mathcal{V}}}\mu\left(\mathcal{D}_1^{\mathcal{V}} f\right) + \frac{1}{\gamma_0^{\mathcal{C}_0}}\mu\left(\mathbb{1}_{\partial\mathcal{C}_0 \text{ is empty}} \mathcal{D}^{\mathcal{C}_0} f\right)\right] \\
&\leq \frac{1}{\gamma^b}\max\left\{\frac{1}{\gamma_1^{\mathcal{V}}}, \frac{1}{\gamma_0^{\mathcal{C}_0}}\right\}\mathcal{D}_H f.
\end{aligned}$$

□

We are left with estimating $\gamma_1^{\mathcal{V}}$ and $\gamma_0^{\mathcal{C}_0}$.

Claim 3.5.4. There exists $C > 0$ such that $\gamma_1^{\mathcal{V}} \geq C q^3$.

Proof. The Dirichlet form $\mathcal{D}_1^{\mathcal{V}}$ is dominated by the Dirichlet form of FA1f on $\mathbb{Z}_+$, and that dynamics has spectral gap which is proportional to q^3 (see [**19**]). □

For $\gamma_0^{\mathcal{C}_0}$ we will use the bisection method, comparing the gap on a box to that of a smaller box. For $L \in \mathbb{N}$, let $\mathcal{L}_L^{\mathrm{NE}}$ be the generator of the north-east dynamics in the box $[L]^2$ with empty boundary (for the north east model this is equivalent to putting empty boundary only above and to the right). Denote its gap by $\gamma_{[L]^2}^{\mathrm{NE}}$. By monotonicity we can restrict the discussion to this dynamics, i.e.,

$$\gamma_0^{\mathcal{C}_0} \geq \gamma_{\mathrm{diam}\,\mathcal{C}_0}^{\mathrm{NE}}. \tag{3.5.1}$$

We will now bound γ^{NE} (see also Theorem 6.16 of [**19**]).

Claim 3.5.5. $\gamma_{[L]^2}^{\mathrm{NE}} \geq e^{3\log q\, L}$.

PROOF. We will prove the result for $L_k = 2^k$ by induction on k. Then monotonicity will complete the argument for all L. Consider the box $[L_k]^2$, and divide it in two rectangles – $R_- = [L_{k-1}] \times [L_k]$ and $R_+ = [L_{k-1}+1, L_k] \times [L_k]$. We will run the following block dynamics

$$\mathcal{L}^{b\mathrm{NE}} f = \left(\mu_{R_+} f - f\right) + \mathbb{1}_{\partial_- R_+ \text{ is empty}} \left(\mu_{R_-} f - f\right),$$

where $\partial_- R_+$ is the inner left boundary of R_+. Again, by Proposition 4.4 of [**19**],

$$\begin{aligned} \operatorname{gap}\left(\mathcal{L}^{b\mathrm{NE}}\right) &= 1 - \sqrt{1 - \mu\left(\mathbb{1}_{\partial_- R_+ \text{ is empty}}\right)} \\ &= 1 - \sqrt{1 - q^{L_k}}. \end{aligned}$$

Therefore for every local function f

$$\begin{aligned} \operatorname{Var} f &\leq \frac{1}{1 - \sqrt{1 - q^{L_k}}} \mathcal{D}^{b\mathrm{NE}} f \\ &= \frac{1}{1 - \sqrt{1 - q^{L_k}}} \mu\left(\operatorname{Var}_{R_+} f + \mathbb{1}_{\partial_- R_+ \text{ is empty}} \operatorname{Var}_{R_-} f\right) \\ &\leq \frac{1}{1 - \sqrt{1 - q^{L_k}}} \mu\left(\frac{1}{\gamma^{\mathrm{NE}}_{R_+}} \mathcal{D}^{\mathrm{NE}}_{R_+} f + \frac{1}{\gamma^{\mathrm{NE}}_{R_-}} \mathcal{D}^{\mathrm{NE}}_{R_-} f\right), \end{aligned}$$

where $\gamma^{\mathrm{NE}}_R, \mathcal{D}^{\mathrm{NE}}_R$ are the spectral gap and Dirichlet form of the north-east dynamics in R with empty boundary conditions for any fixed rectangle R. We see that

$$\gamma^{\mathrm{NE}}_{[L_k]^2} \geq \left(1 - \sqrt{1 - q^{L_k}}\right) \gamma^{\mathrm{NE}}_{[L_{k-1}] \times [L_k]}.$$

If we repeat the same argument dividing $[L_{k-1}] \times [L_k]$ into the rectangles $[L_{k-1}] \times [L_{k-1}]$ and $[L_{k-1}] \times [L_{k-1}+1, L_k]$, we obtain

$$\gamma^{\mathrm{NE}}_{L_{k-1} \times L_k} \geq \left(1 - \sqrt{1 - q^{L_{k-1}}}\right) \gamma^{\mathrm{NE}}_{[L_{k-1}]^2}.$$

Hence,

$$\log \gamma^{\mathrm{NE}}_{[L_k]^2} \geq \log \gamma^{\mathrm{NE}}_{[L_{k-1}]^2} + 2^k \log q - \log 4,$$

yielding

$$\log \gamma^{\mathrm{NE}}_{[L_k]^2} \geq \log q \sum_{n=1}^{k} 2^n - k \log 4$$

which finishes the proof. □

We can now put everything together. Let L be the diameter of $\mathcal{C}_0$. By the second part of Claim 2.2.12

$$\begin{aligned} \tau &\leq \frac{1+q}{q} \frac{1}{\gamma^r} \leq \frac{1+q}{q} \frac{1}{\left(1 - \sqrt{1 - q^{|\mathcal{C}|}}\right) \min\left\{\gamma^{\mathcal{V}}_1, \gamma^{\mathcal{C}_0}_0\right\}} \\ &\leq q^{-4L-1}. \end{aligned} \tag{3.5.2}$$

Finally, we will use the sharpness of the phase transition for the site percolation on the dual graph (see [**1, 22**]):

CLAIM 3.5.6. There exists a positive constant c_2 that depends on π such that $\nu\left(L \geq D\right) \leq e^{-c_2 D}$ for any $D \in \mathbb{N}$.

Using this claim and equation (3.5.2)

$$\begin{aligned}\nu\left(\tau \geq t\right) &\leq \nu\left(q^{-4L-1} \geq t\right) = \nu\left(L \geq \frac{\log t}{4\log\frac{1}{q}} - \frac{1}{4}\right) \\ &\leq C\, t^{c/\log q}.\end{aligned}$$

□

3.6. Fredrickson-Andersen 1 spin facilitated model on the polluted $\mathbb{Z}^2$

In this section we will analyze the KCM defined in equation (3.1.5). Rather than considering the time τ_0, we will ask at which time scale correlation is lost. In the worst case, as explained in section 2.1, this is the relaxation time. We will see, however, that in a typical case this time is shorter.

We will study here the loss of correlation in the occupation of the origin. Clearly, if the origin is surrounded by absent sites it will not change, so we will only consider the case in which π is small enough, and that the origin is in the infinite cluster of present sites. The following theorem shows that the relaxation time scales as q^{-3}, but already at times proportional to $q^{-5/2}$ the correlation becomes small.

THEOREM 3.6.1. *Consider the FA1f model on the infinite cluster of present sites, and assume it contains the origin.*

(1) There exists $c > 0$ such that the relaxation time is bigger than cq^{-3}.

(2) Fix $a > 0$. There exists ν-a.s. q_0 such that for all $q \leq q_0$

$$\left|\mathbb{E}_\mu\left[\eta_0(0)\eta_0(t)\right] - \mu\left(\eta_0\right)^2\right| \leq q\left(1-q\right)\, 2e^{-Ca},$$

where $t = aq^{-5/2}\,\left(\log {}^1\!/_q\right)^{3/2}$.

REMARK 3.6.2. For the case $\pi = 0$, i.e., the FA1f model on $\mathbb{Z}^2$, the relaxation time scales like q^{-2} (up to log corrections). We expect this to be the scale at which correlation is lost whenever the origin belong to the infinite present cluster, but a proof will require more precise knowledge on the structure of this cluster. Still, the theorem above gives us a time scale which is faster than the spectral gap.

3.6.1. The spectral gap (proof of part 1 of Theorem 3.6.1). The gap of the model on polluted $\mathbb{Z}^2$ is at most the gap on $\mathbb{Z}$ – the percolation cluster contains paths of arbitrary length containing only vertices of degree 2, so any test function on $\mathbb{Z}$ could be shifted to such a path giving the same value of $\frac{\mathcal{D}f}{\mathrm{Var} f}$. By [**19**] the spectral gap of the FA1f model on $\mathbb{Z}$ is at most cq^3. □

3.6.2. Loss of correlation (proof of part 2 of Theorem 3.6.1). The proof will be based on the methods of [**43**], by introducing a long-range constraint on the graph.

During this prove γ and C will denote generic positive constants that may change from one line to the other. They are allowed to depend on π but not on q.

Let H be an arbitrary directed graph with no oriented cycles. The orientation of the edges defines a partial order, where for two vertices x, y we say that $x \le y$ if there is a path leading from x to y.

Definition 3.6.3. $B_x(R)$ is the ball of radius R around a vertex x with respect to the graph distance (ignoring orientation). The forward ball is defined as

$$\overrightarrow{B}_x(R) = B_x(R) \cap \{y : y \ge x\}.$$

Definition 3.6.4. Let $V \in \mathbb{N}$ and a vertex x of G. The critical length at x with respect to V is

$$l_x^V = \min\left\{l : \left|\overrightarrow{B}_x(l)\right| \ge V\right\}.$$

Definition 3.6.5. Fix two scaling functions $l_c : \mathbb{N} \to \mathbb{R}$, $L : \mathbb{N} \to \mathbb{R}$ ($L \gg l_c$) and a vertex x_0. We say that H has an l_c forward growth in the domain $B_{x_0}(L)$ if for all $V \in \mathbb{N}$ big enough and for all $x \in B_{x_0}(L(V))$

$$l_x^V \le l_c(V).$$

In other words, l_c forward growth means that all vertices in $B_{x_0}(L(V))$ have at least V sites in their forward ball of radius l_c.

We now focus on the infinite cluster G of present sites, and assume that π is small. Fix

$$\begin{aligned} l_c &= (\alpha+\beta)\sqrt{V}, \\ L &= Ce^{\gamma V}, \end{aligned} \tag{3.6.1}$$

for α, β big enough and C, γ small enough that will be determined later.

Proposition 3.6.6. *Assume that there exists an infinite up-right path starting at 0. Then there exists an orientation of G, for which G has an l_c forward growth in the domain $B_0(L)$.*

We will postpone the proof of this proposition to the end of the section, and continue using the orientation that it provides.

Proposition 3.6.7. *Let $\mathcal{L}_L^0$ be the generator of the FA1f dynamics on $B_0(L)$ with free boundary conditions, for $V = \frac{2\log 1/q}{q}$. Then the gap of $\mathcal{L}_L^0$ is at least $Cq^{5/2}(\log 1/q)^{-3/2}$.*

The strategy of the proof will pass through a long range dynamics on $B_0(L)$:

$$c_x^{(l)} = \begin{cases} 1 & \overrightarrow{B}_x(l) \text{ contains an emtpy site} \\ 1 & \|x\|_1 = L \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{L}^{(l)}$ the generator of the associated KCM on $B_0(L)$ and by $\mathcal{D}^{(l)}$ its Dirichlet form.

CLAIM 3.6.8. There exists ν-a.s. $q_0 > 0$ such that for all $q < q_0$ the spectral gap of $\mathcal{L}^{(l)}$ is greater than $\frac{1}{4}$, choosing $V = \frac{2\log 1/q}{q}$ and $l = l_c(V)$.

PROOF. By choosing q_0 small enough, $l_x^V \leq l$ for all $x \in B_0(l)$. That is, $\vec{B}_x(l)$ contains at least V sites, which means that

$$\mu\left(c_x^{(l)}\right) \geq 1 - (1-q)^V.$$

Then we use Theorem 2 of [**43**]. This theorem says that in a KCM such as the one we consider, with oriented constraints, the spectral gap is greater than $\frac{1}{4}$ if the constraint is likely enough compared to its range. In our case this condition translates to

$$\sup_{z\in B_0(L)} \sum_{x:z\in\overrightarrow{B}_x(l_c)} \mu\left(1 - c_x^{(l)}\right) \leq \frac{1}{4}.$$

Since for every given z the number of vertices x such that $z \in \overrightarrow{B}_x(l_c)$ is bounded by l_c^2, the proof is finished once noting that

$$l_c^2\,(1-q)^V \leq (\alpha+\beta)^2\, V e^{-qV} = (\alpha+\beta)^2\, q \log 1/q,$$

which will be smaller than $\frac{1}{4}$ for q_0 small enough. ☐

In order to move from the long range dynamics to the short range dynamics we use a path argument.

LEMMA 3.6.9. *Fix l, and let $x \in B_0(L)$ and η such that $c_x^{(l)}(\eta) = 1$. Then there exists a path $\eta_0, \dots, \eta_N$ of configurations and a sequence of sites $x_0, \dots x_{N-1}$ such that*

(1) $\eta_0 = \eta$, $\eta_N = \eta^x$.
(2) $\eta_{i+1} = \eta_i^{x_i}$.
(3) $c_{x_i}(\eta_i) = 1$.
(4) $N < 2l$.
(5) $\max_i \#\{\text{empty sites of } \eta_i \text{ besides } x_i\} \leq \#\{\text{empty sites of } \eta_0\} + 1$.
(6) Each flip occurs at most once, i.e., if $x_i = x_j$ and $\eta_i = \eta_j$, then $i = j$.
(7) Let $\xi_x(\eta)$ be the first empty site in $\overrightarrow{B}_x(l)$ for the configuration η according to the order induced by the edge orientation (and an arbitrary order in case of ambiguity). Then for all i either $\xi_x(\eta) = \xi_{x_i}(\eta_i)$ or $\xi_x(\eta) = \xi_{x_i}(\eta_i^y)$ for a vertex y such that there is an edge from x_i to y.

(8) For any $x, x' \in G$ and any configuration η' there exist at most six possible configurations η such that (η', x') is compatible with (η, x). Compatibility means that $\eta' = \eta_i$ and $x' = x_i$ for some $i < N$, where η_i and x_i are the ith configuration and ith flip in the path defined for η and x.

PROOF. We use a path similar to the one defined in [**19**, Theorem 6.4]. $c_x^{(l)}(\eta) = 1$, so we can find an oriented path $z_0, \dots, z_t$ such that $z_0 = x$, $z_t = \xi_x(\eta)$ and $t \le l$. Assume first $\eta_x = 1$. Then we start by flipping z_{t-1}, then z_{t-2}, then flip back z_{t-1}, flip z_{t-3}, flip back z_{t-4} and so on, until we flip z_0. Then we flip z_1 again in order to fill it, and finish.

If, on the other hand, $\eta_x = 0$, we start by flipping z_1, then z_0, then z_2, then z_1 and so on, until flipping z_{t-1}, then z_{t-2}, and then z_{t-1} again.

Verifying conditions 1-7 is immediate. For condition 8 we note that if we know who $\xi_x(\eta)$ is and the initial occupation of x, we can reconstruct the initial configuration by setting all sites between x and ξ_x to occupied. Property 6 guarantees that at most three values of $\xi_x(\eta)$ are possible, and the initial state of x could be either 0 or 1. □

PROOF OF PROPOSITION 3.6.7. Fix $V = \frac{2\log 1/q}{q}$ and $l = l_c(V)$ and L as in equation (3.6.1). Recall equations 2.1.3 and 2.1.5. Using the path defined in Lemma 3.6.9 and denoting $p = 1-q$

$$\begin{aligned}\mathrm{Var} f &\le 4\sum_x c_x^{(l)}\mu(\mathrm{Var}_x f) = 4pq\sum_\eta \mu(\eta)\sum_x c_x^{(l)} c_{x_i}(\eta_i)\left(\sum_{i=1}^N \nabla_{x_i} f(\eta_i)\right)^2 \\ &\le 8pql\sum_\eta\sum_x \mu(\eta)\sum_{x'}\sum_{\eta'} \mathbb{1}_{(\eta',x')\text{ compatible with }(\eta,x)}\, c_{x'}(\eta')\left(\nabla_{x'} f(\eta')\right)^2 \\ &= 8l\sum_{x'}\sum_{\eta'} R\left(\eta'^{x'},\eta'\right)\left(\nabla_{x'} f(\eta')\right)^2 \sum_\eta\sum_x \frac{pq\mu(\eta)}{R(\eta'^{x'},\eta')}\mathbb{1}_{(\eta',x')\text{ compatible with }(\eta,x)}.\end{aligned}$$

$c_{x'}(\eta') = 1$, so R is given by

$$R\left(\eta'^{x'},\eta'\right) = pq\prod_{z\in B_0(L)\setminus\{x'\}} \mu_z(\eta'_z).$$

Since $\mu(\eta) = \prod_{z\in B_0(L)} \mu_z(\eta_z)$, the ratio

$$\frac{pq\mu(\eta)}{R(\eta'^{x'},\eta')} = \mu(\eta_{x'})\prod_{z\ne x':\eta_z\ne\eta'_z} \frac{\mu(\eta_z)}{\mu(\eta'_z)} \le q^{-1}.$$

Hence, noting that $x \in B_l(x')$,

$$\mathrm{Var} f \le 50l^3 q^{-1}\sum_{x'}\sum_{\eta'} R\left(\eta'^{x'},\eta'\right)\left(\nabla_{x'} f(\eta')\right)^2 = 100q^{-1}l^3\mathcal{D}f.$$

This proves Proposition 3.6.7 by inserting $l = (\alpha+\beta)\sqrt{\frac{2\log 1/q}{q}}$ (recall the variational definition of the spectral gap in equation (2.1.6)). □

We can now prove Theorem 3.6.1 using a finite speed of propagation argument, coupling the process generated by $\mathcal{L}$ with that generated by $\mathcal{L}_L^0$. The coupling will be via the graphical construction – we take the same initial conditions and same clock rings with the same coin tosses, so the only difference is in the constraint on the boundary of $B_0(L)$. η denotes the process that evolves according to $\mathcal{L}$, and η^0 the process evolving according to $\mathcal{L}_L^0$.

Recall that $t = aq^{-5/2}\left(\log {}^1/_q\right)^{3/2}$.

Definition 3.6.10. Let B be the event, that there exist a sequence of times $0 \le t_1 \le \cdots \le t_k \le t$ and of sites $x_1, \ldots, x_k \in B_0(L)$ such that x_1 is on the boundary of $B_0(L)$, $x_k = 0$, and the clock of site x_i rang at time t_i for all $i \le k$.

Observation 3.6.11. In the coupling defined above, whenever B^c occurs $\eta_0(t) = \eta_0^0(t)$.

We can now estimate the correlation:

$$\begin{aligned}\mathbb{E}_\mu\left[\eta_0(0)\,\eta_0(t)\right] &= \mathbb{E}_\mu\left[\eta_0(0)\,\eta_0(t)\,\mathbb{1}_B\right] + \mathbb{E}_\mu\left[\eta_0(0)\,\eta_0(t)\,\mathbb{1}_{B^c}\right]\\ &= \mathbb{E}_\mu\left[\eta_0(0)\,\eta_0(t)\,\mathbb{1}_B\right] + \mathbb{E}_\mu\left[\eta_0^0(0)\,\eta_0^0(t)\,\mathbb{1}_{B^c}\right]\\ &= \mathbb{E}_\mu\left[\eta_0(0)\,\eta_0(t)\,\mathbb{1}_B\right] - \mathbb{E}_\mu\left[\eta_0^0(0)\,\eta_0^0(t)\,\mathbb{1}_B\right] + \mathbb{E}_\mu\left[\eta_0^0(0)\,\eta_0^0(t)\right],\end{aligned}$$

thus

$$\left|\mathbb{E}_\mu\left[\eta_0(0)\,\eta_0(t)\right] - \mathbb{E}_\mu\left[\eta_0^0(0)\,\eta_0^0(t)\right]\right| \le 2\mathbb{P}(B).$$

Using the spectral gap of $\mathcal{L}_L^0$ we know that

$$\mathbb{E}_\mu\left[\eta_0^0(0)\,\eta_0^0(t) - (1-q)^2\right] \le q(1-q)\,e^{-Cq^{5/2}\left(\log {}^1/_q\right)^{-3/2} t} = q(1-q)\,e^{-Ca}.$$

The probability of B could be bounded by the possible number of paths $x_1, \ldots, x_k$, times the probability that a Poisson random variable of parameter t will exceed k:

$$\mathbb{P}(B) \le \sum_{k=L}^{\infty} 4^k \sum_{m\ge k} \frac{t^m e^{-t}}{m!} \le e^{-t} \sum_{k=L}^{\infty}\sum_{m\ge k} \frac{(4t)^m}{m!}.$$

By equation (3.6.1) and the choice $V = \frac{2\log {}^1/_q}{q}$, for q small enough $L \ge 20t$, so $\left(\frac{(4t)^m}{m!}\right) / \left(\frac{(4t)^{m+1}}{(m+1)!}\right) = \frac{m+1}{4t} \ge 5$. That is,

$$\sum_{k=L}^{\infty}\sum_{m\ge k} \frac{(4t)^m}{m!} \le \sum_{k=L}^{\infty}\sum_{n\ge 0} \frac{(4t)^k}{k!}\frac{1}{5^n} \le 2\sum_{k=L}^{\infty} \frac{(4t)^k}{k!} \le \sum_{k=0}^{\infty} \frac{(4t)^L}{L!}\frac{2}{5^k} \le 4\frac{(4t)^L}{L!} \le 1.$$

In order to finish the proof of Theorem 3.6.1, consider x_0 in the infinite present cluster, but not necessarily in the infinite oriented cluster. Since G has an l_c forward growth in the domain $B_{x_0}(L - \|x_0\|_1)$, the above estimations hold also for the correlation of $\mathbb{E}_\mu\left[\eta_{x_0}(0)\,\eta_{x_0}(t)\right]$.

We are left with the proof of Proposition 3.6.6.

Definition 3.6.12. G' is the up-right percolating cluster of G, i.e., the sub-graph induced by the vertices connected to infinity by an up-right path. Note that orienting all edges up or to the right gives a directed graph with no oriented cycles.

CLAIM 3.6.13. Assume $0 \in G'$. Fix $\alpha \in \mathbb{R}$ big enough, and let $l_c(V) = \alpha\sqrt{V}$, $L(V) = Ce^{\gamma V}$ for C, γ small enough. Then almost surely G' has an l_c growth in the domain $B_0(L)$.

PROOF. Fix V, $x \in G'$. We ask what is the probability that $l_x^V > l_c(V)$. In other words, we look for the probability that $\left|\overrightarrow{B}_x(l_c)\right| < \frac{1}{\alpha^2} l_c^2$. By **[23]**

$$\nu\left[\frac{1}{l_c^2}\left|\overrightarrow{B}_x(l_c)\right| < \frac{1}{\alpha^2}\right] < Ce^{-\gamma l_c^2}.$$

Therefore

$$\nu\left[\exists x \in B_0(L) \text{ such that } l_x^V > l_c(V)\right] < 4L^2\, Ce^{-\gamma l_c^2} = Ce^{-\gamma V}.$$

The result follows by the Borel-Cantelli lemma. □

In order to extend this result to the entire G, we will use the uniqueness of the infinite cluster of the oriented percolation, that will imply the following observation.

OBSERVATION 3.6.14. $G \setminus G'$ is a union of disjoint finite connected clusters.

Call the set of these cluster $\mathcal{C}$. For $L \in \mathbb{N}$, let

$$\mathcal{C}_L = \{G \in \mathcal{C} \,|\, G \cap B_0(L) \neq \emptyset\}.$$

CLAIM 3.6.15. Fix $\beta \in \mathbb{R}$, and let $D(V) = \beta\sqrt{V}$ and $L = Ce^{\gamma D}$ for γ small enough. Then ν-a.s. for all $V \in \mathbb{N}$ large enough

$$\max_{H \in \mathcal{C}_{L(V)}} \operatorname{diam} H \leq D(V).$$

where C is a positive constant.

PROOF. Fix V. By **[1, 22]**, for every $x_1 \in \mathbb{Z}^2$, the probability that its cluster in $G \setminus G'$ is contained in $x_1 + [-D/2, D/2]$ is exponentially close to 1. In order for a graph $H \in \mathcal{C}$ to have a diameter larger than D it must contain such a vertex. Thus

$$\nu\left[\max_{H \in \mathcal{C}_{L(V)}} \operatorname{diam} H > D(V)\right] < C\,(L + D)^2\, e^{-\gamma D},$$

and the result follows by the Borel-Cantelli lemma. □

PROOF OF PROPOSITION 3.6.6. In G' take the usual up-right orientation. Then, for each $H \in \mathcal{C}$, we take any arbitrary orientation such that all vertices point to G via an oriented path (this could always be done for finite H). We use Claim 3.6.15 and Claim 3.6.13 with α, β given in equation (3.6.1). Fix $x \in B_0(L)$. By Claim 3.6.15, the forward ball $\overrightarrow{B}_x(l_c)$ contains $\overrightarrow{B}_{x'}(\alpha\sqrt{V})$ for some $x' \in G'$. This ball alone contains more than V vertices by Claim 3.6.13, which concludes the proof. □

3.7. Fredrickson-Andersen 2 spin facilitated model on polluted $\mathbb{Z}^2$

Consider now the model defined in equation (3.1.6). The bootstrap percolation was studied in [**31**], where it is shown that when q and π tend to 0, if $\frac{\pi}{q^2}$ is small enough the probability that the origin could be emptied tends to 1, but if $\frac{\pi}{q^2}$ is big this probability tends to 0.

In this section we will study τ_0 of the KCM. We will assume $\pi < q^{2+\varepsilon}$ for some $\varepsilon > 0$, which will mean that the origin is likely to be emptiable. Even in this case the KCM is not ergodic – for any fixed ω there exists with probability 1 a square around the origin whose four corners are absent. The μ-probability that this square is entirely occupied is nonzero (though very small for typical ω), and when it is entirely occupied it could never change. This is very different from the previous models that we have analyzed, and we will have to take into account that τ_0 could be infinite.

THEOREM 3.7.1. *Fix $\varepsilon > 0$. Then for q small enough, with probability tending to* 1 *(with respect to the measure ν) the environment ω is such that*

$$\mathbb{P}_\mu\left(\tau_0 > e^{q^{-1-\varepsilon}}\right) < 4q^{\varepsilon/12}.$$

REMARK 3.7.2. The corresponding lower bound is left out for brevity, but since the bootstrap percolation is monotone, the time it would take to empty the origin is of order at least that of the non-polluted lattice, namely $e^{c/q}$, which will then also bound the emptying time for the KCM. Thus (up to ε) the correct scaling is indeed $e^{c/q}$.

3.7.1. Proof of Theorem 3.7.1. The idea of the proof is to find a "good" event that happens with very high probability. Then use the results of section 2.3 in order to show that not too much time was spent in this good event before τ_0. This, however, is only possible if τ_0 is small, since the fraction of time spent in the good event equals its probability, which is very high.

We start by fixing two scales:

$$\begin{aligned} L &= q^{-1-\varepsilon/3}, \\ l &= q^{-L-1}; \end{aligned} \tag{3.7.1}$$

and defining a good square (which is not the same as Definition 3.2.4, though they play a similar role).

DEFINITION 3.7.3. A square (that is, a subset of $\mathbb{Z}^2$ of the form $x+[L]^2$) is *good* if it contains at least one empty site in each line and in each column, and if none of its sites is absent.

CLAIM 3.7.4. For q small enough $\nu \otimes \mu\left([L]^2 \text{ is good}\right) \geq 1 - 2q^{\varepsilon/3}$.

PROOF. The probability that one of the sites of $[L]^2$ is absent is at most $L^2\pi$, which is bounded by $q^{-2-2\varepsilon/3}\,q^{2+\varepsilon} = q^{\varepsilon/3}$. The probability that one of the line or columns of $[L]^2$ is entirely occupied is at most $2L\,(1-q)^L$, which is asymptotically equivalent to $2q^{-1-\varepsilon/3}e^{-q^{-\varepsilon/3}}$.

This bound tends to 0 much faster than $q^{\varepsilon/3}$, and the union bound given the proof of the claim. □

Together with results from percolation theory (e.g. [**22, 32**, Theorem 1.33]) this implies the following corollary.

COROLLARY 3.7.5. *The $\nu \otimes \mu$-probability that the origin belongs to an infinite cluster of good boxes is at least* $1 - 16q^{\varepsilon/3}$.

DEFINITION 3.7.6. Consider a path of good boxes. We say that the path is *super-good* if one of its boxes contains an empty line.

CLAIM 3.7.7. Fix a self avoiding path of boxes whose length is l. Then

$$\nu \otimes \mu\,(\text{path is super-good} \mid \text{path is good}) \geq 1 - e^{-1/q}.$$

PROOF. Since the events {the path is good} and {one of the boxes contains an empty line} are both increasing we can use the FKG inequality, and bound this probability by the probability that a length l path of boxes (not necessarily good) does not contain an empty line. This conclude the proof, since

$$\left(1 - q^L\right)^l \leq e^{-q^L q^{-L-1}}.$$

□

CLAIM 3.7.8. For q small enough $\nu \otimes \mu\,(0 \text{ belongs to a super-good path of length } l) \geq 1 - 25q^{\varepsilon/3}$.

PROOF. By Corollary 3.7.5 the origin belongs to an infinite cluster of good boxes with probability greater than $1 - 16q^{\varepsilon/3}$. In particular, it is contained in a self-avoiding path of length l. Then we use Claim 3.7.7 and the union bound to conclude. □

DEFINITION 3.7.9. $p_{\mathrm{SG}}\,(\omega)$ is the μ-probability that the origin is contained in a super-good path of length l.

DEFINITION 3.7.10. We say that ω is *low pollution* if $p_{\mathrm{SG}}\,(\omega) > 1 - 5q^{\varepsilon/6}$.

CLAIM 3.7.11. $\nu\,(\text{low pollution}) \geq 1 - 5q^{\varepsilon/6}$.

PROOF. By Claim 3.7.8, $\nu\,(p_{\mathrm{SG}}) \geq 1 - 25q^{\varepsilon/3}$. Since $p_{\mathrm{SG}} \leq 1$, Markov inequality will give the result. □

From now on we think of a fixed ω. Let

$$E = \{0 \text{ belongs to a super-good path of length } l\}\,,$$
$$A = \{\eta_0 = 0\}\,.$$

We will use a path argument in order to bound $\mu\left(T_A^E \mathbb{1}_E\right)$ (recall Definition 2.3.1).

LEMMA 3.7.12. *Fix $\eta \in E$. Then there exists a path $\eta_0, \ldots, \eta_N$ of configurations and a sequence of sites $x_0, \ldots x_{N-1}$ such that*

(1) $\eta_0 = \eta$,
(2) $\eta_N \in A$,
(3) $\eta_{i+1} = \eta_i^{x_i}$,
(4) $c_{x_i}(\eta_i) = 1$,
(5) $N \leq 4L^2 l$,
(6) For all $i \leq N$, η_i differs from η at at most $3L$ points, contained in at most two neighboring boxes.

PROOF. This path is constructed in the exact same manner as Lemma 3.3.8 (see also figures 3.3.1 and 3.3.2). The empty line that exists somewhere along the path could be propagated, until reaching the origin. The only difference between this path and that of Lemma 3.3.8 is that the sites that used to be easy in the proof of Lemma 3.3.8 are now empty (which could only help). □

CLAIM 3.7.13. $\mathbb{E}_\mu\left(T_A^E \mathbb{1}_E(\eta(0))\right) \leq q^{-13L}$

PROOF. Recall equation (2.3.2). Thus

$$\mathbb{E}_\mu\left(T_A^E \mathbb{1}_E(\eta(0))\right) = \mu\left(\mathbb{E}_\eta\left(T_A^E\right)\mathbb{1}_E(\eta)\right) = \mu\left(T_A^E \mathbb{1}_E\right),$$

and the the statement of the claim is equivalent to $\mu\left(T_A^E \mathbb{1}_E\right) \leq q^{-10L}$. We use the path constructed above. Note first that $T_A^E(\eta_N) = 0$, so for $\eta \in E$

$$T_A^E(\eta) = \sum_{i=0}^{N-1} c_{x_i}(\eta_i)\, \nabla_{x_i} T_A^E(\eta_i).$$

Then for q small enough

$$\begin{aligned}
\mu\left(T_A^E \mathbb{1}_E\right)^2 &\leq \mu\left(\left(T_A^E \mathbb{1}_E\right)^2\right) = \sum_{\eta \in E} \mu(\eta) \left(\sum_{i=0}^{N-1} c_{x_i}(\eta_i)\, \nabla_{x_i} T_A^E(\eta_i)\right)^2 \\
&\leq \sum_{\eta \in E} \mu(\eta)\ N \sum_{i=0}^{N-1} c_{x_i}(\eta_i)\ \left(\nabla_{x_i} T_A^E(\eta_i)\right)^2 \\
&= \sum_{\eta \in E} \sum_{i=0}^{N-1} \sum_{\eta'} \sum_{x} \mathbb{1}_{\eta' = \eta_i} \mathbb{1}_{x = x_i} \frac{\mu(\eta)}{\mu(\eta')} \mu(\eta')\ N c_x(\eta')\ \left(\nabla_x T_A^E(\eta')\right)^2 \\
&\leq 4L^2 l\, q^{-3L} \sum_{\eta'} \sum_{x} \sum_{\eta \in E} \sum_{i=0}^{N-1} \mathbb{1}_{\eta' = \eta_i} \mathbb{1}_{x = x_i} \mu(\eta')\ c_x(\eta')\ \left(\nabla_x T_A^E(\eta')\right)^2 \\
&\leq 4L^2 l\, q^{-3L}\, L^2 l\ \left(3L^2\right)^{3L} 4L^2 l \sum_{\eta'} \mu(\eta') \sum_{x} c_x(\eta')\ \left(\nabla_x T_A^E(\eta')\right)^2 \\
&\leq q^{-13L} \mathcal{D}\left(T_A^E\right) = q^{-13L} \mu\left(T_A^E \mathbb{1}_E\right),
\end{aligned}$$

where the last equality is due to Corollary 2.3.2. □

We are ready to prove Theorem 3.7.1. Assume ω is low pollution, an event with probability tending to 1 according to Claim 3.7.11. Let $t = \left(1 - 2q^{\varepsilon/12}\right) q^{-14L}$. By Claim 3.7.13 and Markov's inequality

$$\mathbb{P}_\mu\left(T_A^E \mathbb{1}_E\left(\eta\left(0\right)\right) \geq t\right) \leq 2q.$$

Being low pollution means $\mathbb{P}_\mu\left(\eta\left(0\right) \in E\right) \geq 1 - 5q^{\varepsilon/6}$, thus

$$\begin{aligned}\mathbb{P}_\mu\left(T_A^E \geq t\right) &= \mathbb{P}_\mu\left(T_A^E \mathbb{1}_E\left(\eta\left(0\right)\right) \geq t\right) + \mathbb{P}_\mu\left(T_A^E \mathbb{1}_{E^c}\left(\eta\left(0\right)\right) \geq t\right)\\ &\leq \frac{1}{2}q + \mathbb{P}_\mu\left(\eta\left(0\right) \in E^c\right) \leq 6q^{\varepsilon/6}.\end{aligned}$$

On the other hand, for any $s > 0$,

$$\mathbb{E}_\mu\left(T_s^E\right) = s\mu\left(E\right) \geq \left(1 - 5q^{\varepsilon/6}\right) s,$$

and since $T_s^E \leq s$ we can apply again Markov's inequality (for the positive variable $s - T_s^E$), obtaining

$$\mathbb{P}_\mu\left(T_s^E \leq \left(1 - 2q^{\varepsilon/12}\right) s\right) \leq 3q^{\varepsilon/12}.$$

Combining these inequalities for $s = q^{-14L}$ yields

$$\begin{aligned}\mathbb{P}_\mu\left(\tau_A \geq s\right) &\leq \mathbb{P}_\mu\left(T_{\tau_A}^E \geq T_s^E\right) \leq \mathbb{P}_\mu\left(T_{\tau_A}^E \geq t\right) + \mathbb{P}_\mu\left(T_s^E \leq t\right)\\ &\leq 6q^{\varepsilon/6} + 3q^{\varepsilon/12}.\end{aligned}$$

□

CHAPTER 4

Models on the Galton-Watson tree

While in chapter 3 the constraints were random, in this chapter we will consider a random graph G, which will be a Galton-Watson tree. Section 4.2 is based on [**46**] and discusses the bootstrap percolation, and Section 4.3 presents some preliminary results on the KCM.

4.1. Model, notation, and preliminary results

In this chapter we will discuss the bootstrap percolation and the FA model on a random graph G, chosen according to the Galton-Watson (GW) measure – start at the root (denoted 0), and at each step give each of the leaves a random number of children, independently according to a distribution ξ. We fix a threshold r and assume that ξ_k (the probability to have k children) equals 0 for $k < r$.

Once G is chosen, we fix the following constraint

$$c_x = \begin{cases} 1 & x \text{ has at least } r \text{ empty children} \\ 0 & \text{otherwise} \end{cases}, \tag{4.1.1}$$

and consider the dynamics described in equation (1.3.1).

In [**16**] it is shown that, depending on ξ, the model could have a continuous or a discontinuous phase transition, and an explicit expression for the critical probability q_c is found. Here we will study the time scales of the model, first for the bootstrap percolation and then for the KCM.

4.2. Metastability of the bootstrap percolation

When studying the bootstrap percolation it will be convenient to use the notation ϕ_t^G for the probability that root is occupied at time t (so in particular $\phi_0^G = 1 - q$). This probability random, since G is random, and we will denote its expectation by ϕ_t^ξ.

One particular case, that has been studied in [**8, 12, 25, 21**], is the case of a rooted $(d+1)$-regular tree, or equivalently $\xi_k = \mathbb{1}_{k=d}$. Here, one can find ϕ_t^d recursively using the relation

$$\phi_{t+1}^d = h_d\left(\phi_t^d\right); \tag{4.2.1}$$

$$h_d(x) = (1-q)\,\mathbb{P}\left[\text{Bin}\left(d, 1-x\right) \le r-1\right]. \tag{4.2.2}$$

For the GW tree, such a recursion still holds for the expected value ϕ_t^ξ:

$$\phi_{t+1}^\xi = h_\xi\left(\phi_t^\xi\right); \tag{4.2.3}$$

$$h_\xi(x) = \sum_{k=r}^{\infty} \xi_k h_k(x). \tag{4.2.4}$$

The relation in equation (4.2.3) allows us to find the expected value of ϕ_t^G, but for a specific realization of G ϕ_t^G may differ from that value. For example, fixing t, there is a nonzero probability that a finite neighborhood of the root will have many vertices of high degree, which will result in a smaller ϕ_t^G. However, we will see that ϕ_t^ξ describes almost surely another observable – the density, i.e., the limiting fraction of occupied vertices.

First, denote by $B(R)$ the ball of radius R around the root. We can then define the R-density at time t as

$$\rho_R(t) = \frac{|\{\text{occpied vertices in } B(R) \text{ at time } t\}|}{|B(R)|}.$$

It is natural to expect $\rho_R(t)$ to be close to ϕ_t^ξ, and this is indeed the case, as shown in the following proposition:

Proposition 4.2.1. *Fix t. Then $\lim_{R\to\infty}\rho_R(t) = \phi_t^\xi$ almost surely (in both the graph and the initial state measures).*

We would like to understand the critical behavior of the bootstrap percolation, so following **[8, 16]**, we define the critical probability

$$q_c = \inf_{[0,1]}\left\{q : \phi_\infty^\xi = 0\right\}. \tag{4.2.5}$$

In order to analyze this criticality, define

$$g_k(x) = \frac{h_k(x)}{(1-q)\,x}, \tag{4.2.6}$$

$$g_\xi(x) = \frac{h_\xi(x)}{(1-q)\,x}. \tag{4.2.7}$$

In **[16]**, the following fact is shown:

Fact 4.2.2. *Fix ξ. Then:*

(1) For a given q, ϕ_∞^ξ is the maximal solution in $[0,1]$ of the equation $g_\xi(x) = \frac{1}{1-q}$, and 0 if no such solution exists.
(2) $q_c = 1 - \frac{1}{\max_{[0,1]} g_\xi(x)}$.

We will consider here the behavior near criticality, at q slightly smaller than q_c.

Definition 4.2.3. For $0 < x < 1$ and some positive δ, the δ-entrance time of x is

$$\tau_{x,\delta}^-(q) = \min\{t : \phi_t^\xi < x + \delta\},$$

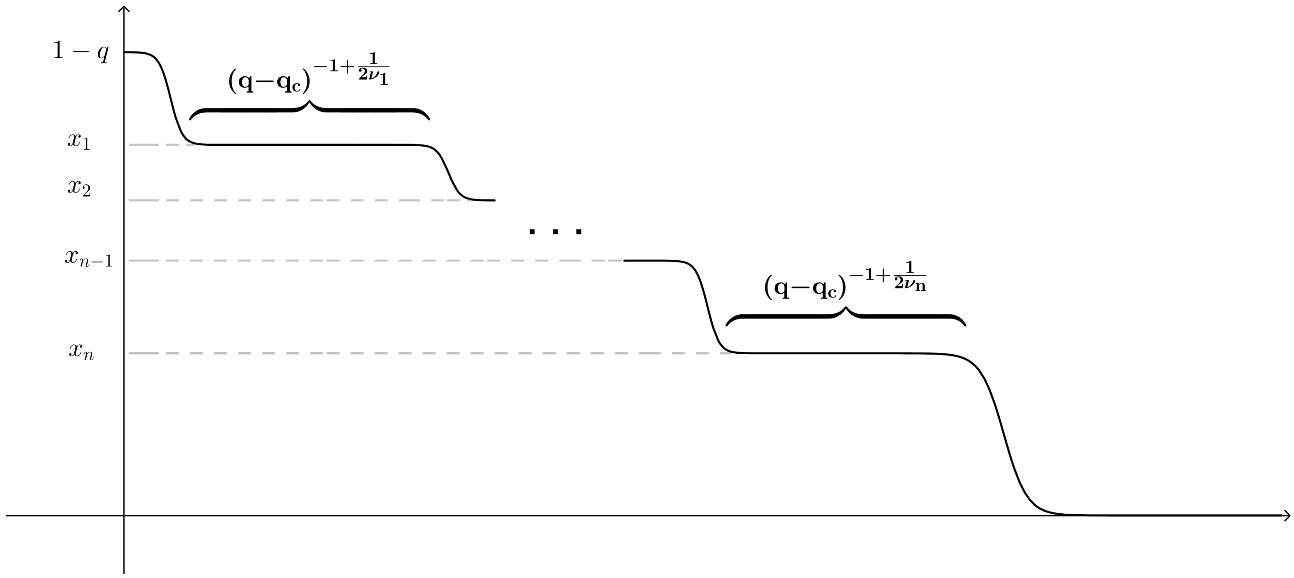

FIGURE 4.2.1. A schematic picture of ϕ_t^ξ as a function of t for a $(\nu_1, \ldots, \nu_n)$-metastable criticality at $x_1 > \cdots > x_n$.

and the δ-exit time is defined as

$$\tau_{x,\delta}^{+}(q) = \min\{t : \phi_t^\xi < x - \delta\}.$$

DEFINITION 4.2.4. Fix $\delta > 0$. We say that the critical point is δ-$(\nu_1, \ldots, \nu_n)$-metastable at $x_1 > \cdots > x_n > 0$ if, for $q \searrow q_c$, the following hold:

(1) $\tau_{x_1,\delta}^{-} = O(1)$.
(2) $\frac{\log\left(\tau_{x_i,\delta}^{+} - \tau_{x_i,\delta}^{-}\right)}{\log(q-q_c)} \xrightarrow{q \searrow q_c} -1 + \frac{1}{2\nu_i}$ for $i = 1, \ldots, n$.
(3) $\tau_{x_{i+1},\delta}^{-} - \tau_{x_i,\delta}^{+} = O(1)$ for $i = 1, \ldots, n$ and $x_{n+1} = 0$.

We say that the critical point is $(\nu_1, \ldots, \nu_n)$-metastable at $x_1 > \cdots > x_n$ if it is δ-$(\nu_1, \ldots, \nu_n)$-metastable at $x_1 > \cdots > x_n$ for small enough δ. See figure 4.2.1.

The following theorem gives a full classification of the metastability properties:

THEOREM 4.2.5. *Fix ξ. Then the metastable behavior is determined by one of the following cases:*

Case 1. *g_ξ attains its maximum at 1. In this case the critical probability is 1.*

Case 2. *g_ξ has a unique maximum at 0. In this case the phase transition is continuous. At the critical point*

$$\frac{\log(\phi_t^\xi)}{\log t} \xrightarrow{t\to\infty} -\frac{1}{\nu}, \tag{4.2.8}$$

where ν is determined by the asymptotic expansion $g_\xi(x) = \frac{1}{q_c} - Cx^\nu + o(x^\nu)$.

Case 3. *The maximum of g_ξ is attained at the points $x_1, \ldots, x_n$ for $1 > x_1 > \cdots > x_n > 0$, and possibly also at 0. In this case the phase transition is discontinuous. For $i =$*

$1,\dots,n$ we may write around x_i

$$g_\xi(x) = \frac{1}{1-q_c} - C_i (x-x_i)^{2\nu_i} + o\left((x-x_i)^{2\nu_i}\right) \tag{4.2.9}$$

for some $C_i > 0$. Then the critical point is $(\nu_1,\dots,\nu_n)$-metastable at $x_1 > \cdots > x_n$.

Remark 4.2.6. In the first case, where the critical probability is 1, it is not clear whether or not an asymptotic expansion exists, since g_ξ is not guaranteed to be analytic. When it does exist, one can recover a result similar to Case 3.

We can now state our main result, proving that all the different metastable behaviors described above can be attained. Actually, the proof of Theorem 4.2.7 is constructive: we provide for any choice of the widths of the multiple plateaus, an offspring distribution which realizes the corresponding metastable behavior.

Theorem 4.2.7.

(1) Let $\nu \in \mathbb{N}$. Then there exists ξ such that the phase transition is continuous, and satisfies equation (4.2.8) at criticality.

(2) Let $(\nu_1,\dots,\nu_n) \in \mathbb{N}^n$. Then there exist ξ and $x_1 > \cdots > x_n$ such that the critical point is $(\nu_1,\dots,\nu_n)$-metastable at $x_1 > \cdots > x_n$.

This behavior is much richer than the dynamics on the regular tree – in that case, if the transition is continuous it satisfies equation (4.2.8) with $\nu = 1$, and when it is discontinuous it could only be 1-metastable at some point x_1.

4.2.1. Proof of Proposition 4.2.1. The idea of the proof is to notice that the main contribution to the density comes from the sites close to the boundary, and then use their independence. Thus, we fix a width w, and consider

$$\rho_{R,w}(t) = \frac{|\{\text{occupied vertices in } B(R)\setminus B(R-w) \text{ at time } t\}|}{|B(R)\setminus B(R-w)|}.$$

First, we claim that $\rho_R(t)$ is approximated by $\rho_{R,w}(t)$ for large w. More accurately, we have $|B(R-w)| \le 2^{-w}|B(R)|$, which also implies that the number of occupied vertices in $B(R)\setminus B(R-w)$ is the same as the number of occupied vertices in $B(R)$, up to a correction of order $2^{-w}|B(R)|$. Thus

$$\rho_R(t) = \rho_{R,w}(t) + O\left(2^{-w}\right). \tag{4.2.10}$$

We would now like to bound the distance between $\rho_{R,w}(t)$ and $1-\phi_t^\xi$. Let $\varepsilon > 0$, and, by equation (4.2.10), take w big enough such that $|\rho_R(t) - \rho_{R,w}(t)| < \frac{\varepsilon}{2}$ uniformly in R. Note that $\rho_{R,w}(t)$ is a weighted average of the w random variables $\rho_{R,1}(t), \rho_{R-1,1}(t), \dots, \rho_{R-w+1,1}(t)$, and consider one of these variables, $\rho_{r,1}(t)$. This variable is the average of the random variables $\mathbb{1}_{v \text{ is empty}}$ for all vertices v of distance r from the root, and since these are independent Bernoulli random variables with mean ϕ_t^ξ, and since there are at least 2^{R-w+1} such variables, we can use a large deviation estimate, yielding

$$\mathbb{P}\left[\left|\rho_{r,1}(t)-\phi_t^{\xi}\right|>\frac{\varepsilon}{2}\right]\leq e^{-c\,2^{R-w+1}}$$

for a positive c that only depends on ε and on ϕ_t^{ξ}. Since for ϕ_t^{ξ} to be far from $\rho_{R,w}(t)$ it must be far from at least one of the variables $\rho_{R,1}(t), \rho_{R-1,1}(t), \ldots, \rho_{R-w+1,1}(t)$, we have

$$\mathbb{P}\left[\left|\rho_{R,w}(t)-\phi_t^{\xi}\right|>\frac{\varepsilon}{2}\right]\leq we^{-c\,2^{R-w+1}}. \tag{4.2.11}$$

Hence, $\rho_R(t)$ is ε-close to ϕ_t^{ξ} with probability larger than $1-we^{-c\,2^{R-w+1}}$, which concludes the proof by the Borel-Cantelli lemma. □

4.2.2. Proof of Theorem 4.2.5. We start with a couple of small results.

CLAIM 4.2.8. g_k is a polynomial of degree $k-1$, whose lowest degree monomial is of degree $k-r$.

PROOF. By equations 4.2.6 and 4.2.2

$$\begin{aligned} g_k(x) &= \frac{\mathbb{P}\left[\mathrm{Bin}(k,1-x)\leq r-1\right]}{x} \\ &= \sum_{i=0}^{r-1}\binom{k}{i}(1-x)^i x^{k-i-1}; \end{aligned}$$

therefore all monomials are of degree between $k-r$ and $k-1$. The coefficient of x^{k-r} is $\binom{k}{r-1}\neq 0$, and the coefficient of x^{k-1} is $\sum_{i=0}^{r-1}\binom{k}{i}(-1)^i$, which is also nonzero since $0<r-1<k$. This concludes the proof. □

CLAIM 4.2.9. $g_r(x), \ldots, g_m(x), x^{m-r+1}, \ldots, x^{m-1}$ is a basis of the linear space of polynomials of degree smaller or equal to $m-1$.

PROOF. Denote $v_1(x)=g_r(x), \ldots, v_{m-r+1}(x)=g_m(x), v_{m-r+2}(x)=x^{m-r+1}, v_m(x)=x^{m-1}$. By Claim 4.2.8, all v's are of degree smaller or equal to $m-1$. Moreover, the matrix whose (i,j) entry is the coefficient of x^j in the polynomial v_i is upper triangular, with nonzero diagonal. This shows that $\{v_i\}_{i=1}^m$ is indeed a basis. □

We will also use Claim 3.9 of [**16**]:

CLAIM 4.2.10. For $\xi_k=\frac{r-1}{k(k-1)}$, $g_\xi(x)=1$.

We are now ready to prove Theorem 4.2.5.

First, we note that $g_k(1)=1$ for all k, so in particular the series $\sum_{k=r}^{\infty}\xi_k g_k(x)$ converges at $x=1$. By Claim 4.2.8, the monomials of degree up to n of the partial sum $\sum_{k=r}^{N}\xi_k g_k(x)$ are fixed once $N>n+r$. From these two facts we conclude that $g_\xi(x)$ is analytic in $(-1,1)$ and continuous at 1. Thus, cases 1, 2 and 3 exhaust all possibilities.

The result will then follow from general arguments of dynamical systems near a bifurcation point. Since the exact calculations are a bit tedious, we only give here a short sketch of the argument, referring to the appendix for the complete proof.

For case 2, the expression

$$\phi_{t+1} = \phi_t - C\left(1-q_c\right)\phi_t^{\nu+1} + o(\phi_t^{\nu+1})$$

could be estimated by comparing to the differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -C\left(1-q_c\right)\phi_t^{\nu+1}.$$

This equation could be solved explicitly, yielding the asymptotics of equation (4.2.8).

For case 3, the approximate differential equation will be

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{x_i}{1-q_c}(q-q_c) - C_i\left(1-q_c\right)x_i(\phi - x_i)^{2\nu_i}.$$

The solution of this equation has a plateau around x_i, whose length diverges as $(q-q_c)^{-1+\frac{1}{2\nu_i}}$. □

4.2.3. Proof of Theorem 4.2.7. For the first part, it will be enough to show that there exist an offspring distribution ξ and a polynomial $Q(x) = b_0 + \cdots + b_{r-2}x^{r-2}$ such that

(1) $g_\xi\left(x\right) = \mathrm{Const} - x^\nu Q(x)$.
(2) $Q(x) > 0$ for all $x \in [0,1]$.

This ξ, according to Theorem 4.2.5 and the fact that $b_0 > 0$, will indeed satisfy equation (4.2.8). Rather than ξ, it will be easier to find a sequence $\{\chi_k\}_{k=r}^{\infty}$ with a finite sum together with a polynomial $P(x) = a_0 + \cdots + a_{r-2}x^{r-2}$, such that

(1) $g_\chi\left(x\right) = \sum_k \chi_k g_k\left(x\right) = 1 - x^\nu P\left(x\right)$.
(2) $\chi_k \geq 0$.
(3) $P\left(x\right) > 0$ for all $x \in [0,1]$.

Taking $\xi = \frac{1}{\sum \chi_k}\chi_k$ will then conclude the proof. Let

$$\chi_k = \begin{cases} \frac{r-1}{k(k-1)} & r \leq k \leq \nu + r - 1 \\ 0 & k \geq \nu + r. \end{cases} \tag{4.2.12}$$

Using Claim 4.2.10 we may write

$$g_\chi\left(x\right) = 1 - \sum_{k=\nu+r}^{\infty} \frac{r-1}{k\left(k-1\right)} g_k\left(x\right).$$

By Claim 4.2.8 g_χ is a polynomial of degree $\nu + r - 2$, therefore $\sum_{k=\nu+r}^{\infty} \frac{r-1}{k(k-1)} g_k\left(x\right)$ equals a polynomial of degree $\nu + r - 2$. Using again Claim 4.2.8, we can define the polynomial

$$P\left(x\right) = \sum_{k=\nu+r}^{\infty} \frac{r-1}{k\left(k-1\right)} \frac{g_k\left(x\right)}{x^\nu}.$$

It is left to show that $P\left(x\right) > 0$ for all $x \in [0,1]$. By equations 4.2.6 and 4.2.2, $P(x)$ is non-negative and could only vanish at $x = 0$. But by Claim 4.2.8, $P(0) = \frac{r-1}{(\nu+r)(\nu+r-1)}\left(\frac{g_{\nu+r}(x)}{x^\nu}\right)_{x=0} \neq 0$. This concludes the first part.

Remark 4.2.11. Note that, by Claim 4.2.9, we can define the projection Pr from the space of polynomials of degree at most $r+\nu-2$ to its subspace spanned by $x^{\nu}, \ldots, x^{\nu+r-2}$ with kernel spanned by $g_r(x), \ldots, g_{\nu+r-1}(x)$. Define also M_0 to be the map from the space of polynomials of degree at most $r-2$ to the space of polynomials of degree at most $r+\nu-2$ given by the multiplication by x^{ν}. Then the first of the conditions above can be written as

$$\mathrm{Pr} M_0 \, P = \mathrm{Pr}\ 1.$$

Since $\mathrm{Pr} \circ M_0$ is bijective, this equation has a unique solution; and what we have shown in the proof is that this solution satisfies the necessary positivity conditions.

We will now prove the second part of the theorem. In analogy with the first one, we will find $\bar{\xi}$, $\overline{Q}(x) = \bar{b}_0 + \cdots + \bar{b}_{r-2}x^{r-2}$ and $x_1 > \cdots > x_n$ such that

(1) $g_{\bar{\xi}}(x) = \overline{\mathrm{Const}} - (x-x_1)^{2\nu_1} \ldots (x-x_n)^{2\nu_n}\, \overline{Q}(x)$.
(2) $\overline{Q}(x) > 0$ for all $x \in [0,1]$.

Similarly to the previous part, we will look for $\{\bar{\chi}_k\}_{k=r}^{\nu+r-1}$ and $\overline{P}(x) = \bar{a}_0 + \cdots + \bar{a}_{r-2}x^{r-2}$ satisfying:

(1) $g_{\bar{\chi}}(x) = \sum_k \bar{\chi}_k g_k(x) = 1 - (x-x_1)^{2\nu_1} \ldots (x-x_n)^{2\nu_n}\, \overline{P}(x)$.
(2) $\bar{\chi}_k > 0$.
(3) $\overline{P}(x) > 0$ for all $x \in [0,1]$.

Note that choosing $\nu = 2\nu_1 + \cdots + 2\nu_n$, χ_k (defined in equation (4.2.12)) is strictly positive for $r \leq k \leq \nu + r - 1$. Since P was required to be strictly positive, we may hope that also after adding a small perturbation $(x_1, \ldots, x_n)$ around 0 there still exists a positive solution $\overline{P}$. More precisely, let us denote by $M_{x_1,\ldots,x_n}$ the multiplication by $(x-x_1)^{2\nu_1} \ldots (x-x_n)^{2\nu_n}$, acting on the polynomials of degree at most $r-2$. In particular, for $x_1, \ldots, x_n = 0$ this is M_0 defined in 4.2.11. Then, we want to show that the solution of

$$\mathrm{Pr} M_{x_1,\ldots,x_n} \overline{P} = \mathrm{Pr}\ 1$$

satisfies the positivity conditions 2 and 3. By continuity of the determinant, when $(x_1, \ldots, x_n)$ is in a small neighborhood of 0 the operator $\mathrm{Pr} M_{x_1,\ldots,x_n}$ is invertible. Moreover, in an even smaller neighborhood of 0 the polynomial $(\mathrm{Pr} M_{x_1,\ldots,x_n})^{-1}\,\mathrm{Pr} 1$ will satisfy the positivity condition 3 – matrix inversion is continuous, and the set of polynomials satisfying this condition is open and contains $(\mathrm{Pr} M_0)^{-1}\,\mathrm{Pr} 1$ by the first part of the proof. Finally, since coordinate projections of $1-(x-x_1)^{2\nu_1} \ldots (x-x_n)^{2\nu_n}\,(\mathrm{Pr} M_{x_1,\ldots,x_n})^{-1}\,\mathrm{Pr} 1$ with respect to the basis defined in Claim 4.2.9 are continuous in $(x_1, \ldots, x_n)$, and since for $(x_1, \ldots, x_n) = 0$ condition 2 is satisfied, by taking $(x_1, \ldots, x_n)$ in a further smaller neighborhood of 0 we are guaranteed to find a polynomial $\overline{P}$ satisfying the required conditions. □

4.2.4. Some remarks and open questions. The first remark is about other possible discontinuities of ϕ. Consider, for example, $r = 2$ and $\xi_k = \frac{3}{5}\mathbb{1}_{k=2} + \frac{2}{5}\mathbb{1}_{k=5}$. The function

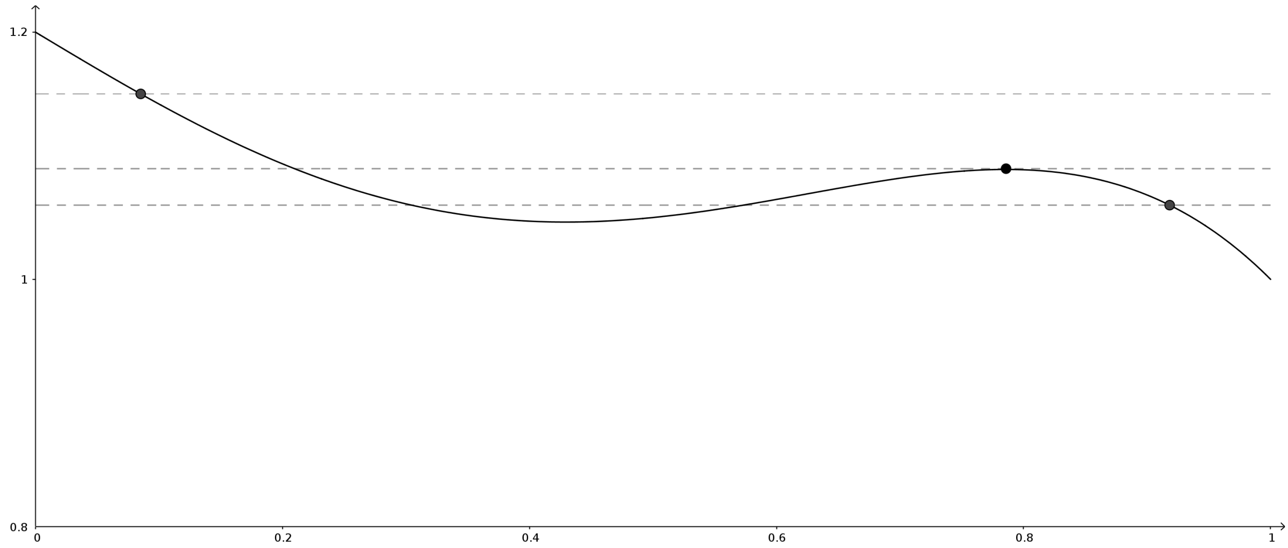

FIGURE 4.2.2. g_ξ for $r=2$ and $\xi_k = \frac{3}{5}\mathbb{1}_{k=2} + \frac{2}{5}\mathbb{1}_{k=5}$. We show three lines $\frac{1}{1-q}$ for three parameters q, intersecting g_ξ at ϕ^ξ_∞. One sees here the discontinuity when $\frac{1}{1-q}$ equals the value of g_ξ at the local maximum.

$g_\xi(x)$ is maximal at $g_\xi(0) = \frac{6}{5}$, then it has a local minimum, followed by a local maximum (see figure 4.2.2).

In this case, recalling Fact 4.2.2, ϕ^ξ_t will have a discontinuity at this local maximum, that is, a second phase transition occurs. We may then expect that one can find ξ giving rise to as many (decreasing) local maxima of g_ξ as we wish:

CONJECTURE 4.2.12. *Let* $\nu^{(1)}_1, \ldots, \nu^{(1)}_{n_1}, \nu^{(2)}_1, \ldots, \nu^{(2)}_{n_2}, \ldots, \nu^{(m)}_{n_m}$. *Then there exists* g_ξ, $\{q_i\}_{i=1}^m$ *and* $\left\{x^{(i)}_j\right\}_{1\le i\le m,\, 1\le j\le n_i}$ *such that* q_i *is a critical point which is* $\left(\nu^{(i)}_1, \ldots, \nu^{(i)}_{n_i}\right)$*-metastable at* $x^{(i)}_1, \ldots, x^{(i)}_{n_i}$.

Another possible phase transition, studied in [**25**] for the case of regular trees, is when infinite empty clusters start to appear, but the density is still bigger than 0. Following the proof of Proposition 3.9 in [**25**], one sees that it applies also for the bootstrap percolation on GW trees, showing that the critical probability $q^{(\infty)}_c$ below which infinite clusters no longer appear is strictly smaller than q_c defined in equation (4.2.5), unless $\xi_k = \mathbb{1}_r$.

Finally, a few open problems remain. First, though we have seen that the metastability above describes the almost sure density of specific realizations G of the tree, we do not have a quenched description of ϕ^G_t. Its value depends highly on the local structure, so it doesn't seem to be close to ϕ^ξ_t that we have analyzed, but we can still ask what is its metastability picture. One easy observation is that by monotonicity $\phi^G_{\tau^-} - \phi^G_{\tau^+}$ is always positive (for $\tau^- \le \tau^+$, as in Definition 4.2.3). Therefore, when $\phi^\xi_{\tau^-} - \phi^\xi_{\tau^+}$ is very small, Markov's inequality says that $\phi^G_{\tau^-} - \phi^G_{\tau^+}$ is small with high probability. That is, during a time interval in which ϕ^ξ is flat there will be a high probability for ϕ^G to be flat. This suggests that ϕ^G, though it could be far from ϕ^ξ, should have the same metastability structure. In order to prove this, however, one must also show that between plateaus the value of ϕ^G changes by $\Omega(1)$, which is beyond my current reach.

Other than this specific model, more questions could be studied on the time behavior of the bootstrap percolation on random graphs, such as $G_{n,p}$ or the configuration model, for which the critical point is understood but, to my knowledge, not its metastability properties.

4.3. Fredrickson-Andersen model on a Galton-Watson tree – one example

We will now discuss the spectral gap of the FA model on Galton-Watson trees (equation (4.1.1)). This model is still not well understood, so this section will be more a description of the thought process than the presentation of a complete result.

We focus on one specific example:

$$r = 2,$$
$$\xi_k = \begin{cases} \frac{1}{2} & k = 3 \text{ or } 4 \\ 0 & \text{otherwise.} \end{cases}$$

We start by a few observations that will direct us to the questions we can ask about this model.

OBSERVATION 4.3.1. Let $\mathcal{L}^d$ be the generator of the dynamics on the d-ary tree (i.e., for $\xi_k = \mathbb{1}_{k=d}$). Then

$$\text{gap}\mathcal{L}^3 \leq \text{gap}\mathcal{L} \leq \text{gap}\mathcal{L}^4.$$

PROOF. This is a consequence of the monotonicity of the Dirichlet form. Consider a test function f on our Galton-Watson tree G. We can take G to be a subtree of the 4-ary tree, and the Dirichlet form $\mathcal{D}^4 f$ will be greater than $\mathcal{D}f$. For the other inequality, we can think of the trinary tree as a subtree of G, so $\mathcal{D}^3 f \leq \mathcal{D}f$ for any f defined on it. □

OBSERVATION 4.3.2. Let q_c be the critical probability, starting from which the KCM is ergodic. $q_c^{(3)}$ and $q_c^{(4)}$ will correspond to the dynamics on the 3- and 4-ary trees. Then

$$q_c^{(4)} < q_c < q_c^{(3)}.$$

PROOF. All three probabilities could be calculated explicitly using Fact 4.2.2. The values obtained are

$$q_c^{(3)} = \frac{1}{9},\ q_c^{(4)} = \frac{13}{256},\ q_c = \frac{31\sqrt{31} - 154}{89 + 31\sqrt{31}}.$$

□

When $q < q_c$ the dynamics is not ergodic, so we will only be interested here in $q > q_c$. When $q > q_c^{(3)}$ the spectral gap is positive, since according to [**42**] the spectral gap of $\mathcal{L}^3$ is positive; and their bounds hold.

PROPOSITION 4.3.3. *Fix $q_c < q < q_c^{(3)}$. Then the spectral gap of $\mathcal{L}$ is 0.*

PROOF. We start with a definition.

DEFINITION 4.3.4. Fix $l \in \mathbb{N}$ and a vertex $x \in G$. The subtree of G rooted at x of depth l is called *bad* if each vertex has only 3 children.

The spectral gap of $\mathcal{L}^{(3)}$ is 0, so we can find a local function f on the trinary tree such that $\frac{\mathcal{D}^{(3)} f}{\mathrm{Var} f}$ is as small as we want. G will contain bad subtrees of all depth, so in particular we can define f on G such that it is supported on a bad subtree. This shows that the spectral gap is 0. □

We have thus identified two critical probabilities – q_c in which the model becomes ergodic, and $q_c^{(3)}$ in which a gap is opened. For $q \in \left(q_c, q_c^{(3)}\right)$ the dynamics is ergodic, but with infinite relaxation time.

More refined estimation could be made by considering only the first L levels of the tree. We call this tree G_L, and denote by $\mathcal{L}_L$ the generator of the dynamics on G_L with free boundary conditions. For concreteness we will take $q = 0.1$, which is between q_c and $q_c^{(3)}$.

THEOREM 4.3.5. *Consider $\mathcal{L}_L$ defined above and $q = 0.1$. With probability tending to 1 as L tends to ∞,*

$$L^{-15} \leq gap\left(\mathcal{L}_L\right) \leq L^{-0.09}.$$

Before proving this theorem, note that a more interesting result would be if the spectral gap scaled like $L^{-\alpha}$ for some $\alpha < 1$ – by a finite speed of propagation argument (like in the proof of Theorem 3.6.1) we would obtain a stretched exponential decay of correlation, i.e., for local f with mean 0

$$\mathbb{E}_\mu\left[f\left(\eta\left(0\right)\right) f\left(\eta\left(t\right)\right)\right] \leq C_f e^{-t^{1-\alpha}}.$$

By optimizing the proof, some improvement on the exponent 15 for the lower bound and 0.09 for the upper bound could be achieved; but the method we use does not enable us to find a lower bound that scales with an exponent smaller than 1, nor an upper bound that will exclude this possibility.

4.3.0.1. *Proof of the upper bound.* In order to show that the spectral gap is small, we need to find an appropriate test function. Fix L, and take l such that $L \geq 3^l + l$.

CLAIM 4.3.6. G_L contains a bad subtree of depth l with probability greater than $1 - e^{-\left(\frac{3}{2}\right)^L}$.

PROOF. For a fix vertex, the probability to be the root of a bad subtree of depth l is $\left(\frac{1}{2}\right)^{3^l}$. The number of vertices at level $L - l$ is at least 3^{L-l}. Therefore, the probability that none of them is the root of a bad subtree is $\left[1 - \left(\frac{1}{2}\right)^{3^l}\right]^{3^{L-l}}$. □

Assume now that $\mathbb{T}_3$ is a bad subtree of G with depth l, rooted at r_3. Let f be the indicator of the function, that r_3 is in the span of $\mathbb{T}_3$. First, for l big enough $\phi_l^{(3)}$ approaches its limit, which by Fact 4.2.2 and our choice of q is equal $\frac{5}{6}$. In particular

$$\mathrm{Var} f = \phi_l^{(3)}\left(1 - \phi_l^{(3)}\right) \geq 0.1.$$

Next, we would like to find, for x on the boundary of $\mathbb{T}_d$,

$$\mu\left(c_x \mathrm{Var}_x f\right) \;\leq\; q\,(1-q)\,\mu\left(f\left(\omega^x\right) \neq f\left(\omega\right)\right).$$

LEMMA 4.3.7. *Consider the bootstrap percolation on $\mathbb{T}_3$, and fix x on its boundary. Then the probability that r_3 is in the span of $\mathbb{T}_3$ for η^x but not for η is at most 0.3^l when l large enough. We call this probability π.*

PROOF. Let $x = v_0, \ldots, v_l = r_3$ be the path from x to the root. Assume that the root is in the span of $\mathbb{T}_3$ for η^x but not for η. This means that for every $0 \leq i \leq l$, the vertex v_i is occupied, and exactly one of its two siblings is in the span of $\mathbb{T}_3$. Thus

$$\pi = \prod_{i=0}^{l-1} (1-q)\; 2\phi_i^{(3)} \left(1 - \phi_i^{(3)}\right),$$

and the result follows since, for l big enough, $\phi_i^{(3)}$ is close to $\frac{5}{6}$. □

All that is left is to estimate the Dirichlet form:

$$\mathcal{D}f = \sum_x \mu\left(c_x \mathrm{Var}_x f\right) = \sum_{x \in \partial \mathbb{T}_3} \mu\left(c_x \mathrm{Var}_x f\right) \leq \sum_{x \in \partial \mathbb{T}_d} q\,(1-q)\, 0.3^l$$
$$= 0.09\, 3^l\, 0.3^l \;\leq 0.9^l\, \mathrm{Var} f.$$

The result follows since $L^{-0.09} > L^{\log_3 0.9} = 0.9^{\log_3 L}$. □

4.3.0.2. *Proof of the lower bound.* Fix $l \in \mathbb{N}$, and consider the long range dynamics defined by

$$c_x^{(l)} = \begin{cases} 1 & x \text{ in empty for the bootstrap percolation after } l+1 \text{ steps} \\ 0 & \text{otherwise} \end{cases}.$$

By [**42**], if $\sup_{x \in G} \mu\left(1 - c_x^{(l)}\right) < \frac{1}{l+1}$ then for any f

$$\mathrm{Var} f \;\leq\; 8\mathcal{D}_L^{(l)} f = 8 \sum_x \mu\left(c_x^{(l)} \mathrm{Var}_x f\right). \tag{4.3.1}$$

CLAIM 4.3.8. Let $l = \log_2 L + 71$. Then the (quenched) probability $\nu\left[\sup_{x \in G} \mu\left(1 - c_x^{(l)}\right) \leq \frac{1}{l+1}\right]$ is at most e^{-L}.

PROOF. First, note that for a fixed vertex, the expected value of $\mu\left(1 - c_x^{(l)}\right)$ (with respect to the quenched variables) is ϕ_l^{ξ}. By the recursion formula equation (4.2.3), we can find that, for l big enough, it is bounded by $2e^{-2^{l-70}}$. Therefore, by Markov's inequality,

$$\nu\left[\mu\left(1 - c_x^{(l)}\right) > \frac{1}{l+1}\right] \leq (l+1)\, 2e^{-2^{l-71}}.$$

Then by the union bound

$$\nu\left[\sup_{x \in G} \mu\left(1 - c_x^{(l)}\right) > \frac{1}{l+1}\right] \leq 4^L\, (l+1)\, 2e^{-2^{l-71}}.$$

□

We will now use a path argument in order to bound the spectral gap.

DEFINITION 4.3.9. Let $G_x^{(l)}$ be the tree of depth l rooted at x, and assume that $c_x^{(l)} = 1$. We construct $\mathbb{T}_2$ the emptying tree of x as follows:

(1) Put x in $\mathbb{T}_2$.
(2) Look for a leaf y of $\mathbb{T}_2$ that has at least two children in the span of $G_x^{(l)}$. If there is such y, add to $\mathbb{T}_2$ first two children of y that are in the span of $G_x^{(l)}$ (according to some fixed arbitrary order).
(3) Repeat step 2 until all leaves of $\mathbb{T}_2$ have at most one child in the span of $G_x^{(l)}$.

Note that this construction is possible since $c_x^{(l)} = 1$, and thus at each step $\mathbb{T}_2$ is in the span of $G_x^{(l)}$.

OBSERVATION 4.3.10. Fix x, let $\mathbb{T}_2$ be its emptying tree for the configuration η, and take a leaf y of $\mathbb{T}_2$. Then the emptying tree of x for η^y is also $\mathbb{T}_2$.

CLAIM 4.3.11. Let $G_x^{(l)}$ be the tree of depth l rooted at x, assume that $c_x^{(l)} = 1$, and let $\mathbb{T}_2$ be its emptying tree. Assume that all sites of $\mathbb{T}_2$ other than its leaves are occupied. Then there is a path of configurations $\eta_0, \dots, \eta_N$ and sites $x_0, \dots, x_{N-1} \in \mathbb{T}_2$ such that

(1) $\eta_0 = \eta$ and $\eta_N = \eta^x$.
(2) $\eta_{i+1} = \eta_i^{x_i}$,
(3) $c_{x_i}(\eta_i) = 1$,
(4) $N \le 4^{l+1}$,
(5) η_i differs from η at at most $2l$ sites.

PROOF. Assume that such a path exists for each of the children of x (in $\mathbb{T}_2$), but for $l-1$ rather than l. Then we empty both of them, then empty x, and then flip again both children. The path could thus be constructed by induction. □

DEFINITION 4.3.12. Let $G_x^{(l)}$ be the tree of depth l rooted at x, assume that $c_x^{(l)} = 1$, and let $\mathbb{T}_2$ be its emptying tree. The canonical path $\eta_0, \dots, \eta_N$ is defined by

$$\eta_i = \eta \wedge \overline{\eta}_i,$$

where $\overline{\eta}_0, \dots, \overline{\eta}_N$ is the path constructed in Claim 4.3.11 for the initial state in which all are occupied except for the leaves of $\mathbb{T}_2$ that are empty.

We are now ready to apply the path argument. Fix a local function f and a vertex $x \in G$.

$$\begin{aligned}\mu\left(c_x^{(l)}\mathrm{Var}_x f\right) &= q(1-q)\sum_\eta \mu(\eta)\, c_x^{(l)} \left(\nabla_x f\right)^2 \\ &= q(1-q)\sum_\eta \mu(\eta)\left(\sum_{i=1}^N c_{x_i}(\eta_i)\,\nabla_{x_i} f(\eta_i)\right)^2\end{aligned}$$

$$\leq 4^{l+1} \sum_{\eta} \mu(\eta) \sum_{i=1}^{N} \left(c_{x_i}(\eta_i) \nabla_{x_i} f(\eta_i)\right)^2$$

$$= 4^{l+1} \sum_{\eta'} \mu(\eta') \sum_{x'} \sum_{\eta} \mathbb{1}_{\eta'=\eta_i} \mathbb{1}_{x'=x_i} \sum_{i=1}^{N} \frac{\mu(\eta)}{\mu(\eta')} \left(c_{x'}(\eta') \nabla_{x'} f(\eta')\right)^2 .$$

The ratio $\frac{\mu(\eta)}{\mu(\eta')}$ could be bounded by $\left(\frac{q}{1-q}\right)^{2l}$. Then, if we know η' and x, we can construct $\mathbb{T}_2$. Therefore, if we also know i, η could be one of 2^{2l} configurations. Finally, we can bound $\mathbb{1}_{x'=x_i}$ by 1, and obtain

$$\mu\left(c_x^{(l)} \mathrm{Var}_x f\right) \leq 4^{l+1}\, 10^{2l}\, 2^{2l}\, 4^{l+1} \sum_{x' \in G_x} \sum_{\eta'} \mu(\eta') \left(c_{x'}(\eta') \nabla_{x'} f(\eta')\right)^2 .$$

All that is left is to combine this result with equation (4.3.1):

$$\mathrm{Var} f \leq 8\mathcal{D}_L^{(l)} f = 8 \sum_x \mu\left(c_x^{(l)} \mathrm{Var}_x f\right) \leq 128\, 6400^l\, 4^l \sum_{x' \in G} \sum_{\eta'} \mu(\eta') \left(c_{x'}(\eta') \nabla_{x'} f(\eta')\right)^2$$

$$= C\, L^{\log_2 25600} \mathcal{D} f.$$

□

CHAPTER 5

The Kob-Andersen model on $\mathbb{Z}^d$

This chapter presents a joint work with Fabio Martinelli and Cristina Toninelli, that also appears in [**41**]. It concerns with the Kob-Andersen model, which is a kinetically constrained lattice gas model. Particles are allowed to jump to a nearby empty site, as long as the constraint is satisfied, and it is in a sense the Kawasaki version of the Fredrickson-Andersen model. In [**20**] the relaxation time of the Kob-Andersen model has been studied in dimension 2 and parameter $k = 2$ (to be defined later on). By a simple comparison with the simple exclusion process this relaxation time, when considering the model in a box of side L and free boundary, is at least CL^2. However, [**20**] only show an upper bound of $L^2 (\log L)^4$. In this work we have proven a diffusive (i.e., CL^2) scaling in all dimensions. Moreover, we are able to bound the constant C, showing that its divergence at small q is at most that of the relaxation time in the corresponding Fredrickson-Andersen model.

5.1. The Kob-Andersen model and the main result

Given an integer L, and a parameter $q \in (0, 1)$, we let $\Lambda = [L]^d$

$$\partial\Lambda = \{x \in \Lambda : \exists y \notin \Lambda \text{ with } \|x - y\|_1 = 1\}.$$

and consider the probability space $(\Omega_\Lambda, \mu_\Lambda)$ where

$$\Omega_\Lambda = \left\{\eta \in \{0, 1\}^{\mathbb{Z}^d} \ : \ \eta_x = 0 \text{ for all } x \notin \Lambda\right\}$$

and μ_Λ is the product Bernoulli(1-q) measure. Given $\eta \in \Omega_\Lambda$ and $V \subset \Lambda$, we shall say that V is empty (for η) if $\eta_x = 0 \ \forall x \in V$.

Fix an integer $k \in [2, d]$ and, for any given a pair of nearest neighbor sites x, y in Λ, write $c_{xy}(\cdot)$ for the indicator of the event that both x and y have at least $k - 1$ empty neighbors among their nearest neighbors in Λ without counting x, y

$$c_{xy}(\eta) = \begin{cases} 1 & \text{if } \sum_{z:\|x-z\|_1=1, z\neq y}(1 - \eta_z) \geq k - 1 \text{ and } \sum_{z:\|y-z\|_1=1, z\neq x}(1 - \eta_z) \geq k - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1.1}$$

and set

$$\eta^{xy}_z := \begin{cases} \eta_z & \text{if } z \notin \{x, y\} \\ \eta_x & \text{if } z = y \\ \eta_y & \text{if } z = x. \end{cases}$$

$$\eta_z^x := \begin{cases} \eta_z & \text{if } z \neq x \\ 1-\eta_x & \text{if } z = x. \end{cases}$$

The Kob-Andersen model in Λ with parameter k, for short the KA-kf model, with *constrained exchanges* in Λ and *unconstrained sources* at the boundary $\partial\Lambda$ is the continuous time Markov process defined through the generator which acts on local functions $f : \Omega_\Lambda \to \mathbb{R}$ as

$$\mathcal{L}f(\eta) = \sum_{\substack{x,y\in\Lambda \\ \|x-y\|_1=1}} c_{xy}(\eta)[f(\eta^{xy}) - f(\eta)] + \sum_{x\in\partial\Lambda}[(1-\eta_x)(1-q) + \eta_x q][f(\eta^x) - f(\eta)]. \quad (5.1.2)$$

In words, every pair of nearest neighbors sites x, y such that $c_{xy}(\eta) = 1$, with rate one and independently across the lattice, exchange their states η_x, η_y. In the sequel we will sometimes refer to such a move as a *legal exchange*. Furthermore every boundary site, with rate one and independently from anything else, updates its state by sampling it from the Bernoulli(1-q) measure. Notice that these latter moves are unconstrained and that for $k = 1$ the KA-1f chain coincides with the symmetric simple exclusion in Λ with sources at $\partial\Lambda$. It is easy to check that the KA-kf chain is reversible w.r.t μ_Λ and irreducible thanks to the boundary sources. Let $T_{\rm rel}(q, L)$ be its *relaxation time* i.e. the inverse of the spectral gap in the spectrum of its generator $\mathcal{L}_\Lambda$.

THEOREM 5.1.1. *For any $q \in (0, 1)$ there exists a constant $C(q)$ such that*

$$L^2 \le T_{\rm rel}(q, L) \le C(q)L^2.$$

Moreover, as $q \to 0$ the constant $C(q)$ can be taken equal to

$$C(q) = \begin{cases} \exp_{(k-1)}\left(c/q^{1/(d-k+1)}\right) & \text{if } 3 \le k \le d, \\ \exp(c\log(q)^2/q) & \text{if } k = 2 \le d, \end{cases} \quad (5.1.3)$$

where $\exp_{(r)}$ denotes the r-times iterated exponential and c is a numerical constant.

REMARK 5.1.2. The lower bound in the theorem follows from a simple comparison of the KA-kf chain with the symmetric simple exclusion in Λ and it was already established in [**20**]. An interesting open problem already for $k = 2$ is to prove a diffusive lower bound of the form $C'(q)L^2$ with $C'(q)$ diverging to $+\infty$ as $q \to 0$.

5.2. Proof of the main theorem

The standard variational characterization of the spectral gap of $\mathcal{L}$ implies immediately that the *upper bound* on $T_{\rm rel}(q, L)$ of Theorem 5.1.1 is equivalent to the Poincaré inequality

$$\mathrm{Var}(f) \le C(q)L^2\mathcal{D}(f) \quad \forall\, f : \Omega_\Lambda \mapsto \mathbb{R}, \quad (5.2.1)$$

where $C(q)$ is as (5.1.3), $\mathrm{Var}(f)$ denotes the variance of f w.r.t. the reversible measure μ and $\mathcal{D}(f)$ is the Dirichlet form associated to the generator (5.1.2)

$$\mathcal{D}(f) = \sum_{\substack{x,y\in\Lambda \\ \|x-y\|_1=1}} \mu\Big(c_{xy}(\nabla_{xy}f)^2\Big) + \sum_{x\in\partial\Lambda} \mu\Big(\mathrm{Var}_x(f)\Big), \tag{5.2.2}$$

where $\nabla_{xy}f(\eta) := f(\eta^{xy}) - f(\eta)$ $\mathrm{Var}_x(f)$ is the local variance w.r.t. η_x.

We will prove (5.2.1) in several steps.

The first step consists in proving a coarse-grained constrained Poincaré inequality with long range constraints (see Proposition 5.2.11) under the assumption that the probability $\pi_l(k,d)$ of a certain *good* event (see Definition 5.2.2) is sufficiently large. Here l is the mesoscopic scale characterizing the coarse-grained construction and $2 \le k \le d$ is the parameter of the KA-model. The necessary tools for this part are developed in Sections 5.2.1 and 5.2.2.

The second step (see Section 5.2.4) consists developing canonical flows techniques for the KA model in order to bound from above the r.h.s. of the coarse-grained Poincaré inequality by $C(l,q)(L/l)^2\,\mathcal{D}(f)$, with $C(l,q) \le e^{O(l^{d-1}(|\log(q)|+\log(l)))}$ (see Proposition 5.2.45 and Corollary 5.2.46).

The final step (see Section 5.2.6) proves that it is possible to choose $l = l(q,k,d)$ in such a way that $\pi_l(d,k)$ is large enough and $C(l,q) \le C(q)$ as $q \to 0$, where $C(q)$ is as in (5.1.3).

5.2.1. Coarse graining. Let l be such that $N := L/l \in \mathbb{N}$. W.l.o.g. we also assume that $\frac{1}{2}\sqrt{N} \in \mathbb{N}$. Later on (see Section 5.2.6) we will choose l as a function of q and suitably diverging as $q \to 0$. We will then consider the coarse grained lattice of boxes with side l. We will denote this lattice by $\mathbb{Z}^d_l$. As a graph it is equal $\mathbb{Z}^d$, but its elements represent coarse grained boxes rather than sites. Let also $\Lambda_l = [N]^d \subset \mathbb{Z}^d_l$, where we use the notation $[N]$ for the set $\{1,\dots,N\}$. Vertices of the coarse-grained lattice $\mathbb{Z}^d_l$ will always be denoted using letters $i,j,\dots$ while vertices of the original lattice $\mathbb{Z}^d$ will be denoted $x,y,\dots$. The boundary $\partial\Lambda_l \subset \Lambda_l$ of Λ_l will consists of the vertices of Λ_l with at least one nearest neighbor (in $\mathbb{Z}^d_l$) not in Λ_l. We partition the lattice $\mathbb{Z}^d$ into mesoscopic boxes of side l indexed by vertices in $\mathbb{Z}^d_l$. If $B := [l]^d$ then B_i will denote the box $B + li, i \in \mathbb{Z}^d_l$. In particular $\Lambda = \cup_{i\in\Lambda_l} B_i$. Sometimes we shall simply write "the box i" meaning the box B_i. For $x \in \mathbb{Z}^d$ we denote by $B(x)$ the mesoscopic box containing x.

Definition 5.2.1 (Frameable configurations). Given the d-dimensional cube $\mathcal{C}_n = [n]^d$ and an integer $j \le d$ we define the j^{th}-frame of $\mathcal{C}_n$ as the union of all $(j-1)$-dimensional faces containing the vertex $(1,\dots,1)$ of $\mathcal{C}_n$. Next we introduce the set of (d,j)-*frameable* configurations of $\{0,1\}^{\mathcal{C}_n}$ as those configurations which are connected by legal KA-jf exchanges inside $\mathcal{C}_n$ to a configuration for which the j^{th}-frame of $\mathcal{C}_n$ is empty.

We are finally ready for our definition of a box being good for a given configuration.

DEFINITION 5.2.2 (Good boxes). Given $\eta \in \mathcal{E}_\Lambda$, we say that the box B is (d,k)-good for η if all $(d-1)$-dimensional slices of B parallel to the axes are $(d-1,k-1)$-frameable for all configurations $\eta' \in \mathcal{E}_\Lambda$ that differ from η in at most one site. The probability that the d-dimensional box B is (d,k)-good will be denoted by $\pi_l(d,k)$.

REMARK 5.2.3. For $d=2$, $k=2$ a box is $(2,2)$-good if it contains at least two empty sites in every row and every column.

Notation warning Whenever the value of d,k is clear from the context we shall simply write that a box is good if it is (d,k)-good. We shall also say that a vertex $i \in \mathbb{Z}_l^d$ is (d,k)-good if the box B_i is (d,k)-good.

5.2.2. Tools from oriented percolation. In this section we collect and prove certain technical results from oriented percolation which will be crucial to prove the aforementioned coarse-grained constrained Poincaré inequality. We shall work on the coarse-grained lattice $\mathbb{Z}_l^d$ so that any vertex $i \in \mathbb{Z}_l^d$ is representative of the mesoscopic box B_i in the original lattice $\mathbb{Z}^d$. The main result here is Proposition 5.2.8. Throughout this section the parameters d,k will be kept fixed.

DEFINITION 5.2.4 (Paths). An *up-right* or *oriented* path γ in $\mathbb{Z}_l^d$ starting at i and of length $n \in \mathbb{N}$ is a sequence $\big(\gamma^{(1)},\dots,\gamma^{(n)}\big) \subset \mathbb{Z}_l^d$ such that $\gamma^{(1)} = i$ and $\gamma^{(t+1)} \in \big\{\gamma^{(t)}+\vec{e}_1, \gamma^{(t)}+\vec{e}_2\big\}$ for all $t \in [n-1]$. γ is *focused* if $d_\gamma(t) := d\big(\gamma^{(t)}, \{j :\ j = i + s(\vec{e}_1+\vec{e}_2),\ s \in \mathbb{N}\}\big)$ satisfies $\max_{t\in[n]} d_\gamma(t) \le \sqrt{n}$. Two consecutive elements of γ form an edge of γ and we say that γ,γ' are *edge-disjoint* if they do not share an edge. Finally, we say that γ is *good* if $\gamma^{(t)}$ is good for all $t \in [n]$.

DEFINITION 5.2.5 (Good family of paths). Fix $i \in \mathbb{Z}_l^d$. A family of paths $\mathcal{G}$ is said to form a good family for i if the following conditions hold:

(1) All paths in $\mathcal{G}$ are good up-right focused paths starting at i of length $2N$.
(2) The paths of $\mathcal{G}$ are *almost edge-disjoint* i.e. any common edge is at distance at most $\sqrt{N}$ from i.
(3) $|\mathcal{G}| \ge \frac{1}{2}\sqrt{N}$.

REMARK 5.2.6. Since each vertex can be starting point of at most two edge disjoint paths, the cardinality of $\mathcal{G}$ necessarily satisfies $|\mathcal{G}| \le 2\sqrt{N}$.

Given $i \in \mathbb{Z}_l^d$ let D_i be the segment

$$D_i = \Big\{ i + \Big(\frac{1}{2}\sqrt{N} - t\Big)\vec{e}_1 + \big(\frac{1}{2}\sqrt{N} + t\big)\vec{e}_2 \mid -\frac{1}{2}\sqrt{N} \le t \le \frac{1}{2}\sqrt{N} \Big\}. \tag{5.2.3}$$

Let also $H^{(n)}, V^{(n)}$ be the rectangular subsets of the form

$$H^{(n)} = \{j \in \mathbb{Z}_l^d :\ j = i + a\vec{e}_1 + b\vec{e}_2, a \in [0,l_n], b \in [0,l_{n-1}]\}$$
$$V^{(n)} = \{j \in \mathbb{Z}_l^d :\ j = i + a\vec{e}_1 + b\vec{e}_2, a \in [0,l_{n-1}], b \in [0,l_n]\}$$

where $l_n = 10^n$. We shall prove that the existence of a good family of paths is guaranteed by the simultaneous occurrence of certain events $\mathcal{A}, \mathcal{B}$ and $\{\mathcal{C}_n\}_{n=1}^{n_*}$, where $n_* = \min\{n : \ l_n \geq \sqrt{N}\}$ (cf. Figure 5.2.1).

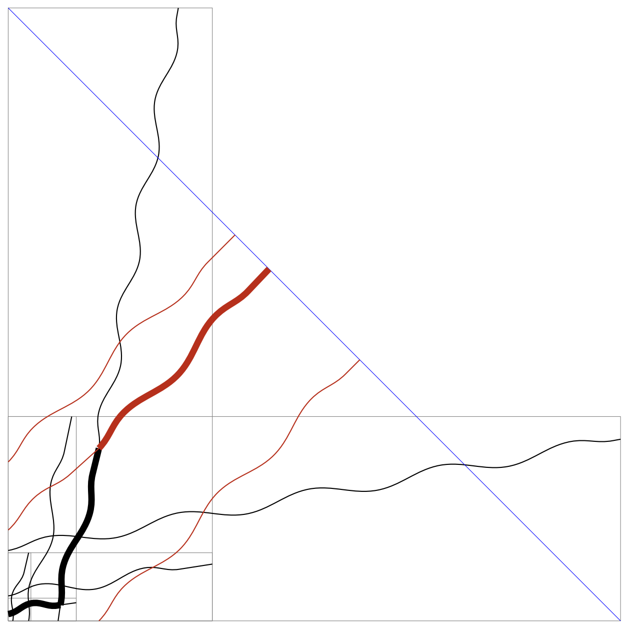

FIGURE 5.2.1. A graphical illustration of the proof of Lemma 5.2.7. For better rendering the drawn paths are not perfectly oriented and the ratio among the sides of rectangles in the drawings is not 1/10 as it should be. The blue segment corresponds to the set D_i. The red paths are the good up-right paths guaranteed by the event $\mathcal{B}$. The blacks paths are the good up-right hard crossings guaranteed by the events C_n.

DEFINITION. (1) Let R_i be the rectangle in $\mathbb{Z}_l^d$ whose short sides are D_i and $D_i + 2N(\vec{e}_1 + \vec{e}_2)$. Then $\mathcal{A}$ is the event that there are at least $1.9\sqrt{N}$ edge-disjoint good up-right paths contained in R_i and connecting D_i with $D_i + 2N(\vec{e}_1 + \vec{e}_2)$.

(2) $\mathcal{B}$ is the event that the set $\cup_{t\in[0,\sqrt{N}]}\{i+t\vec{e}_1\} \cup \{i+t\vec{e}_2\}$ is connected to at least $0.7\sqrt{N}$ vertices of $D_i \setminus (H_{n_*} \cup V_{n_*})$ by a good up-right path,

(3) $\mathcal{C}_n$ is the event that i is good and there exists a good up-right hard-crossing of both $V^{(n)}$ and $H^{(n)}$, i.e. a good up-right path connecting the two short sides of $V_n(H_n)$ and which is contained in $V_n(H_n)$.

LEMMA 5.2.7. *Assume that $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_n$ occurs for all $n \in [n_*]$. Then there exists a good family of paths for i.*

PROOF. We show first that i is connected by a good up-right path to the set $D_i \cap H_{n_*}$ and to the set $D_i \cap V_{n_*}$. Let $n_1 = 1$, and define recursively $n_{k+1}, k \in [n_\star - 1]$ as the largest integer n such that there exists a crossing of $H^{(n \wedge n_*)}$ starting from the set $\{i + t\vec{e}_2, t \in [0, l_{n_k}]\}$. By assumption, the sequence $\{n_k\}_{k=1}^{n}$ is strictly increasing as long as $n_k \leq n_*$. Then, starting from $V^{(1)} \equiv V^{(n_1)}$ we can first follows the lowest hard crossing of $H^{(n_2)}$ until we reach a hard crossing of $V^{(n_2)}$. Then we follow the latter until meeting a hard crossing of $H^{(n_3)}$ and so on. At the end of this procedure the set $V^{(1)}$, and a fortiori the box i, becomes connected by a good up-right path to the right short side of $H_i^{(n_*)}$ and hence also to one of the vertices of $D_i \cap H_{n_*}$. The same construction can be repeated symmetrically by inverting the role of $H_n^{(n)}$ and $V_i^{(n)}$. Therefore we conclude that there exists a good up-right path connecting i to $D_i \cap H_{n_*}$ and a good up-right path connecting i to $D_i \cap V_{n_*}$.

Suppose without loss of generality that $l_{n_*} = \sqrt{N}$. Then, since $|D_i \cap \mathbb{Z}_l^d \cap H_{n_*}| = |D_i \cap \mathbb{Z}_l^d \cap V_{n_*}| = \sqrt{N}/10$ and since each each vertex can be the starting point of at most two edge disjoint paths, event $\mathcal{A}$ guarantees that there are at least $(1.9\sqrt{N} - 2\sqrt{N}/10 - 2\sqrt{N}/10)/2 = 0.75\sqrt{N}$ sites in $D_i \setminus (H_{n_*} \cup V_{n_*})$ that are the starting point of an edge-disjoint up-right paths crossing R_i. Thus, by using event $\mathcal{B}$ and noticing that $|D_i \setminus (H_{n_*} \cup V_{n_*}) \cap \mathbb{Z}_l^d| = 0.8\sqrt{N}$, we get that there are at least $0.65\sqrt{N}$ vertices of $D_i \setminus (H_{n_*} \cup V_{n_*})$ which are at the same time the starting point of edge-disjoint up-right paths crossing R_i and the ending points of up-right paths from $\cup_{t \in [0,\sqrt{N}]} \{i + t\vec{e}_1\} \cup \{i + t\vec{e}_2\}$ to D_i. Using now the fact that i is connected by a good up-right path to the set $D_i \cap H_{n_*}$ and to the set $D_i \cap V_{n_*}$, we conclude that there exist at least $0.65\sqrt{N}$ good up-right paths from i to $D_i + 2N(\vec{e}_1 + \vec{e}_2)$ which, after crossing D_i become edge-disjoint and never leave R_i. The thick path of Fig. 5.2.1 is one of these paths, drawn up to its crossing with D_i (R_i is not depicted in the figure due to lack of space). These paths form the sought good family as required. □

Our next task is to prove that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}_n$ are very likely if $\pi_l := \pi_l(d,k)$ is sufficiently close to one uniformly in n. As proved in Section 5.2.6 that will be the case if the mesoscopic scale l is suitably chosen as a function of q, d, k.

PROPOSITION 5.2.8. *For any $\lambda > 0$ there exists $\pi_* < 1$ such that for $\pi_l \geq \pi_*$ and all $n, N \in \mathbb{N}$*

$$(a) \qquad \mu(\mathcal{C}_n) \geq 1 - e^{-\lambda l_{n-1}},$$
$$(b) \qquad \mu(\mathcal{B}) \geq 1 - e^{-\lambda\sqrt{N}},$$
$$(c) \qquad \mu(\mathcal{A}) \geq 1 - e^{-\lambda\sqrt{N}}.$$

In particular a family of good paths starting at i exists w.h.p if π_l is sufficiently close to one.

PROOF. (a) This can be proven by a contour argument. Consider the rectangle $V^{(n)}$, and assume that it does not contain a good hard crossing. Then consider the path on the dual lattice that forms the upper contour of the set of sites that are connected to the bottom of the rectangle via an up-right good path. Since there is no vertical crossing, this path necessarily takes l_n steps to the right and ends somewhere on the right boundary of $V^{(n)}$. By using the fact that each time this dual path makes a step to the right or downwards, this implies the presence of a bad vertex, it is not difficult to prove that for π_l sufficiently large depending on

λ it holds

$$\mu(\text{there is not a good hard crossing}) = \mu(\mathcal{C}_n^c) \leq e^{-\lambda l_n}$$

(b) Consider the down-left good oriented paths starting from sites of $D_i \setminus (H_{n*} \cup V_{n*})$. The event $\mathcal{B}$ certainly occurs if at least 7/8 of the points in this set are the starting point of an infinite down-left good oriented path. The upper bound on the probability of $\mathcal{B}$ then follows directly from [**23**, Theorem 1][1].

(c) The main tool here is the *max-flow min-cut theorem* (see e.g. [**14**]). For any directed graph (V, E) we consider a capacity function, giving every edge $e \in E$ a positive number $c_e \in \mathbb{R}^+$. For two sets of vertices s and t, we say that a flow from s to t is a positive function of the edges, $f : E \to \mathbb{R}^+$ such that :(i) for all $e \in E$ it holds $f_e \leq c_e$, (ii) for all $v \notin s \cup t$ it holds $\sum_{u:(u,v)\in E} f_{u,v} = \sum_{w:(v,w)\in E} f_{v,w}$, namely for vertices outside $s \cup t$ the incoming flow equals the outgoing flow. The *value of the flow* is defined as the total flow going in t (which is the same as the flow leaving s), namely $\sum_{w\in t} \sum_{v:(v,w)\in E} f_{v,w}$.

A cut (S, T) will be a partition of V in two subsets S and T, such that $s \subseteq S$ and $t \subseteq T$. The *value of the cut* is the the sum of capacities of the edges pointing from S to T.

Theorem 5.2.9. *(Max-Flow Min-Cut theorem) The maximal value of a flow is equal the minimal value of a cut. Moreover, if all capacities are in $\mathbb{Z}$, there is a maximal flow with integer values.*

In order to use this theorem, we first define our graph. The vertex set is

$$V = \left\{ i + a\vec{e}_1 + b\vec{e}_2 \,:\, a, b \in [N]\,,\, a + b \geq \sqrt{N},\, |a - b| \leq \sqrt{N} \right\} \cap \Lambda_l,$$

and the edges are

$$E = \{(j, j') \,:\, j \text{ is good and } j' \in \{j + \vec{e}_1, j + \vec{e}_2\}\}\,.$$

We define s:

$$s = \left\{ j \in V \,:\, \|i - j\|_1 = \sqrt{N} \right\},$$

and t:

$$t = V \cap \{j \,|\, (\vec{e}_1, j) = N \text{ or } (\vec{e}_2, j) = N\}\,.$$

Giving all edges capacity 1, the maximal value of a flow will be the number of edge disjoint paths that we are after. We have thus reduced the problem to the following claim:

Claim 5.2.10. If π_l is large enough, for the graph above with probability greater than $1 - e^{-\lambda\sqrt{N}}$, the value of any cut is at least $1.9\sqrt{N}$.

Proof of the claim. In order to prove the claim, we will construct, for every fixed cut (S, T), a dual path $\gamma^*_{S,T}$ that will separate S from T. We will then show that such a path intersects at least $1.9\sqrt{N}$ edges in E.

[1]Though the Theorem is stated for the contact process, it also holds for oriented percolation as stated in [**23**]).

First, let us define a dual graph V^* for some fixed (S,T). Its vertices will be the faces of Λ_l that have at least three neighbors in V. That is,

$$V^* = \left\{ i^* \in \Lambda_l + \frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2 \,:\, \#\{i \in V \,:\, \|i^* - i\|_1 = 1\} \geq 3 \right\}.$$

□

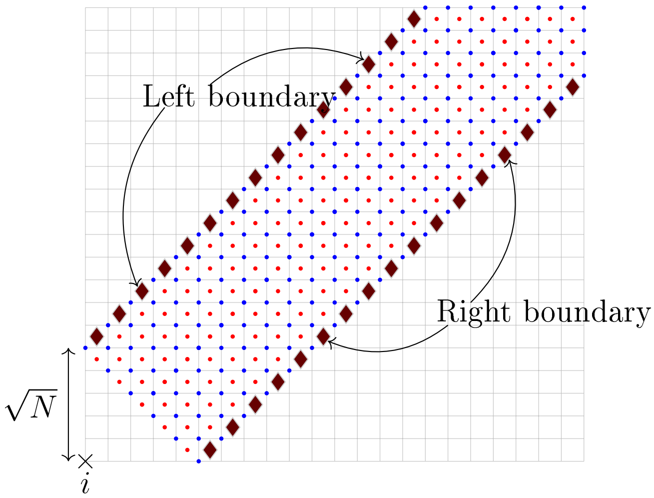

FIGURE 5.2.2. Black dots are the vertices of V, gray dots are the vertices of V^*, diamonds are the left and right boundary of V^*.

PROOF. Its (directed) edges will depend on the cut (S,T). For $i^*, j^* \in V^*$, (i^*, j^*) is an edge if $\|i^* - j^*\|_1 = 1$, and if it has a site of S to its left and a site of T to its right. We will separate the vertices of V^* in three parts:

(1) The right boundary

$$\left\{ i + \left(\sqrt{N} + \frac{1}{2} + a\right)\vec{e}_1 + \left(\frac{1}{2} + a\right)\vec{e}_2 \,:\, a \in [N] \right\} \cup V^*,$$

(2) the left boundary

$$\left\{ i + \left(\sqrt{N} + \frac{1}{2} + a\right)\vec{e}_1 + \left(\frac{1}{2} + a\right)\vec{e}_2 \,:\, a \in [N] \right\} \cup V^*,$$

(3) the interior, which will include all vertices that are neither in the right nor in the left boundary.

Focusing on a fixed vertex $j^* \in V^*$, we can count the edges going into j^* and the edges going out of j^* if we know which of the neighboring vertices of V (namely $\{j \in V \,:\, \|j^* - i\|_1 = 1\}$) are in S and which are in T. By checking all possibilities, one can verify that the incoming degree of a vertex in the interior of V^* equals its outgoing degree (see right part of Fig. 5.2.3).

At the boundaries, however, there could be vertices that have an outgoing degree different from the incoming degree. Consider a site on the right boundary (see left part of Fig. 5.2.3)

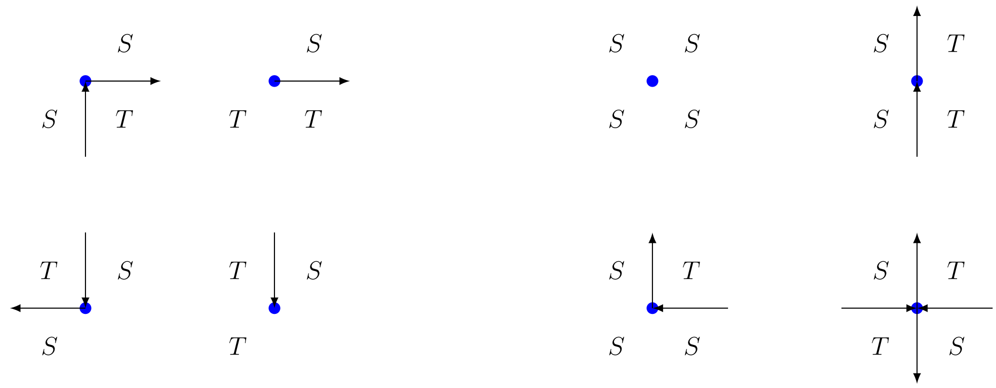

FIGURE 5.2.3. Incoming and outgoing degrees of vertices on the boundary of V^* (left) and its interior (right)

$j_a^* = i + \left(\sqrt{N} + \frac{1}{2} + a\right)\vec{e}_1 + \left(\frac{1}{2} + a\right)\vec{e}_2$. Let $j_a^+ = j_a^* + \frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2$ and $j_a^- = j_a^* - \frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2$. If both j_a^+ and j_a^- are in S, or if both are in T, then the incoming degree of j_a^* is the same as its outgoing degree. However, if $j_a^+ \in S$ and $j_a^- \in T$ then the incoming degree is 1 and the outgoing degree is 0. For the case $j_a^+ \in T$ and $j_a^- \in S$, we have an outgoing degree 1 and incoming degree 0. $j_a^+ = j_{a+1}^-$, therefore the total outgoing degree of sites on the right boundary is

$$\#\left\{a : j_a^- \in S,\, j_{a+1}^- \in T\right\}$$

and the total incoming degree is

$$\#\left\{a : j_a^- \in T,\, j_{a+1}^- \in S\right\}.$$

But since the first site (i.e., j_0^-) is in s and the last site is in t, the incoming degree must be smaller by 1 than the outgoing degree.

By the exact same argument, we can find that the incoming degree of the left boundary is larger by 1 than its outgoing degree. This implies that there exists a dual path $\gamma_* = \left(j_*^{(1)}, \dots, j_*^{(n)}\right)$, where $j_*^{(1)}$ in on the right boundary and $j_*^{(n)}$ is on the left boundary. In particular, $n \geq 2\sqrt{N}$.

The value offspring the cut (S, T) is at least the number of edges in E pointing from S to T and crossing γ_*. Thanks to the choice of the direction of the edges in V^*, this could be written as

$$\#\left\{t : j_*^{(t+1)} - j_*^{(t)} \in \{-\vec{e}_1, \vec{e}_2\} \text{ and } \left(j_*^{(t+1)}, j_*^{(t)}\right) \text{ crosses an edge in } E\right\}.$$

We therefore consider the number of steps that γ_* takes in each direction:

$$\begin{aligned} R &= \#\left\{t : j_*^{(t+1)} - j_*^{(t)} = \vec{e}_1\right\}, \\ L &= \#\left\{t : j_*^{(t+1)} - j_*^{(t)} = -\vec{e}_1\right\}, \\ U &= \#\left\{t : j_*^{(t+1)} - j_*^{(t)} = \vec{e}_2\right\}, \\ D &= \#\left\{t : j_*^{(t+1)} - j_*^{(t)} = -\vec{e}_2\right\}. \end{aligned}$$

Observe that $i_*^{(n)} - i_*^{(1)} = (R-L)\,\vec{e}_1 + (U-D)\,\vec{e}_2$, and since

$$\begin{aligned}\left(\vec{e}_1 - \vec{e}_2, j_*^{(1)}\right) &= (\vec{e}_1 - \vec{e}_2, i) + \sqrt{N},\\ \left(\vec{e}_1 - \vec{e}_2, j_*^{(n)}\right) &= (\vec{e}_1 - \vec{e}_2, i) - \sqrt{N},\end{aligned}$$

$U + L - D - R = 2\sqrt{N}$. Therefore, since $U + L + R + D = n$ we get $U + L = \frac{n}{2} + \sqrt{N}$.

We will consider the erased edges, namely

$$\{(j, j') \,:\, j \text{ is bad and } j' \in \{j + \vec{e}_1, j + \vec{e}_2\}\}\,.$$

Assume now that the value of the cut is less than $1.9\sqrt{N}$, from the previous observations it follows that γ_* must cross at least $U + L - 1.9\sqrt{N} = n/2 - 0.9\sqrt{N}$ erased edges. Since every such erased edge comes from a bad vertex, and since at most two erased edges could share the same bad vertex, at least $n/4 - 0.45\sqrt{N}$ of the vertices to the left of γ_* are bad. Therefore, the probability that there exists a cut with value less than $\leq 1.9\sqrt{N}$ is upper bounded by the probability that there exists a dual path of length $n \geq 2\sqrt{N}$ with at least $n/4 - 0.45\sqrt{N}$ bad vertices on its left. Since there are at most 2^n dual paths of length n, if π_l was taken large enough depending on λ, we get

$$\mu(\text{value of any cut is} \geq 1.9\sqrt{N}) \geq 1 - \sum_{n=2\sqrt{N}}^{\infty} 2^n \sum_{k=n/4-0.45\sqrt{N}}^{n} \binom{n}{k} (1-\pi)^k \geq 1 - e^{-\lambda\sqrt{N}}.$$

□

The proof of the proposition is complete. □

5.2.3. A long range Poincaré inequality. Recall the setting of Sections 5.2.1, 5.2.2 and in particular Definition 5.2.5 of a good family of paths for a vertex $i \in \mathbb{Z}_l^d$. Let $Q_i = i + \{0,1\}^d \setminus \{0\}^d \subset \mathbb{Z}_l^d$ and define

$$\hat{c}_i = \begin{cases} 1 & \text{if any } j \in Q_i \text{ is good and there exists a good family of paths for } i + \vec{e}_1, \\ 0 & \text{otherwise} \end{cases} \tag{5.2.4}$$

In this section we shall prove the following result. Recall that $\pi_l := \pi_l(d,k)$ is the probability that any given $i \in \mathbb{Z}_l^d$ is (d,k)-good.

PROPOSITION 5.2.11. *There exists $\pi_* < 1$ such that for any $\pi_l \geq \pi_*$ and any local function $f : \Omega_\Lambda \to \mathbb{R}$*

$$Var(f) \leq 4 \sum_{i \in \Lambda_l} \mu\Big(\hat{c}_i\, Var_{B_i}(f)\Big). \tag{5.2.5}$$

PROOF. We will closely follow the proof of [**43**, Theorem 2.6]. Let $\tilde{c}_i$ be the indicator of the event $\mathcal{A} \cap \mathcal{B} \cap_{n=1}^{n_*} \mathcal{C}_n$ (see Definition 5.2.2), together with the requirement that Q_i is good. By Lemma 5.2.7 $\tilde{c}_i \leq \hat{c}_i$ for all $i \in \mathbb{Z}_l^d$. Hence, in order to prove (5.2.5), it is enough to prove the stronger constrained Poincaré inequality in which in the r.h.s. of (5.2.5) the constraint $\hat{c}_i$ is replaced by $\tilde{c}_i$. Using Proposition 5.2.8 together with the obvious bound $\mu(Q_i \text{ is good}) \geq$

$1-(2^d-1)(1-\pi_l)$, the proof of the latter is now identical to the one given in [**43**, Theorem 2.6]. □

5.2.4. Constructing the T-step moves. In this section we will construct a set of T-step moves – sequences of at most $T \in \mathbb{N}$ legal moves for the KA dynamics that could be chained together in order to flip the state of an arbitrary point $x \in \mathbb{Z}^d$.

The next definition describes how to move from one configuration to the other using only legal KA exchanges.

DEFINITION 5.2.12. Given a finite connected subset V of Λ and $\mathcal{M} \subset \Omega$, a T-step move $M = (M_0, \dots, M_T)$ taking place in V and with domain $\text{Dom(M)} = \mathcal{M}$ is a function from $\mathcal{M}$ to Ω^{T+1} such that the sequence $M\eta = (M_0\eta, \dots, M_T\eta)\,, \eta \in \mathcal{M}$, satisfies:

(1) $M_0\eta = \eta$,
(2) for any $t \in [T]$, the configurations $M_{t-1}\eta$ and $M_t\eta$ are either identical or linked by a legal KA-transition contained in V.

DEFINITION 5.2.13. Given a T-step move M its *information loss* $\text{Loss}_t(M)$ at time $t \in [T]$ is defined as

$$2^{\text{Loss}_t(M)} = \sup_{\eta' \in \text{Dom}(M)} \#\left\{\eta \in \text{Dom(M)} \,|\, M_t\eta = M_t\eta',\, M_{t+1}\eta = M_{t+1}\eta'\right\}.$$

In other words, knowing $M_t\eta$ and $M_{t+1}\eta$, we are guaranteed that η is one of at most $2^{\text{Loss}_t(M)}$ configurations. We also set $\text{Loss}(M) := \sup_t \text{Loss}_t(M)$. The energy barrier of M is defined as

$$E(M) = \sup_{\eta \in \text{Dom}(M)} \sup_{t \in [T]} \left(\#\left\{\text{empty sites in } M_t\eta\right\} - \#\left\{\text{empty sites in } \eta\right\}\right).$$

The main result is as follows. Recall that $Q_i = i + \{0,1\}^d \setminus \{0\}^d \subset \mathbb{Z}_l^d$ and that $N = L/l$.

PROPOSITION 5.2.14. *Fix $i \in \Lambda_l$ and $x \in B_i$. If $i + \vec{e}_1 + \vec{e}_2 \in \Lambda_l$ fix also an up-right path γ connecting $i + \vec{e}_1 + \vec{e}_2$ to $\partial\Lambda_l$. Then there exists a T-step move M with Dom(M) $=$ $\{\eta \,|\, \gamma$ is good and all $j \in Q_i \cap \Lambda_l$ are good$\}$, taking place in $\cup_{j\in\gamma} B_j \cup (Q_i \cap \Lambda_l)$ and such that, for all $\eta \in$ Dom(M) and all $t \in [T]$, $M_t\eta \in$ Dom(M) and $M_T\eta$ is the configuration η flipped at x. Moreover M can be chosen to satisfy:*

$$T \le CNl^\lambda, \ \left|\mathcal{T}_M^{(j)}\right| \le Cl^\lambda \ \forall j \in \Lambda_l$$

for $k = 1, 2$,

$$T \le CN2^{l^d}, \ \left|\mathcal{T}_M^{(j)}\right| \le C2^{l^d} \ \forall j \in \Lambda_l$$

for $k \ge 3$, and

$$\textit{Loss(M)} \le C\log_2(l)\, l^{d-1}, \ E\,(M) \le Cl^{d-1}$$

for all k, where $\mathcal{T}_M^{(j)}$ is the set of indices $t \in [T]$ such that for some $\eta \in Dom(M)$ the configurations $(M_t\eta, M_{t+1}\eta)$ are linked together by a legal KA-transition inside B_j. The constants C, λ may depend only on k and d.

We start with a few definitions that we will use in this construction of this move.

DEFINITION 5.2.15. Fix two directions $e, e' \in \{\pm e_1, \ldots, \pm e_d\}$. Then the (e, e')-reflection of a site $x \in [l_1] \times \cdots \times [l_d]$ is the site $R_{e,e'}x$ of $[l_1] \times \cdots \times [l_d]$ given by

$$R_{e,e'}x = x - \langle e, x\rangle e - \langle e', x\rangle e' + \langle e, x\rangle e' + \langle e', x\rangle e \mod (l_1, \ldots, l_d)$$

for $e \neq -e'$, and

$$R_{e,e'}x = x - 2\langle e, x\rangle e$$

for $e = -e'$. This is an isometry that sends the vector e to e'.

DEFINITION 5.2.16. Fix $l_1, \ldots, l_d$, and consider the box $B = [l_1] \times \cdots \times [l_d]$. Let $\eta \in \{0,1\}^B$. We define the notion of (k,d)-framed in the directions $\vec{e}_1, \ldots, \vec{e}_d$

Case 1. For $k = 1$ we say that the box is (k,d)-framed if the site $(1, \ldots, 1)$ is empty.

Case 2. For $d = k-1$ the box is (k,d)-framed when it is entirely empty.

Case 3. For $d \geq k > 1$ the box is framed when the first face$\{1\} \times [l_2] \times \cdots \times [l_d]$ is $(k, d-1)$-framed in the directions $\vec{e}_2, \ldots, \vec{e}_d$, and each of the faces $\{m\} \times [l_2] \times \cdots \times [l_d]$, $m = 2, \ldots, d$ is $(k-1, d-1)$-framed in the directions $\vec{e}_2, \ldots, \vec{e}_d$.

Being framed in other directions (given by permutations of $\pm\vec{e}_1, \ldots, \pm\vec{e}_d$) is defined by isometry, that is, B is framed in the directions $\vec{e}_1', \ldots, e_d'$ for the configuration η if $R_{e_1e_1'} \ldots R_{e_de_d'}B$ is framed in the direction $\vec{e}_1, \ldots, \vec{e}_d$ for the configuration η' defined as

$$\eta'(x) = \eta\left(R_{e_1,\zeta e_1}x, \ldots, R_{e_d,\zeta e_d}x\right).$$

EXAMPLE 5.2.17. For $k = 2$, $d = 2$ a box is framed if its bottom line and leftmost column are empty. For $k = 2$, $d = 3$ a box is framed if the three edges touching the box's corner are empty. For $k = 3$,$d = 3$ a box is framed if the three faces touching the corner are empty.

The connection with the frameability defined in section 5.2.1 is given by the following characterization.

PROPOSITION 5.2.18. *The box $[l_1] \times \cdots \times [l_d]$ is (k,d)-framed in the directions $\vec{e}_1, \ldots, \vec{e}_d$ for $\eta \in \{0,1\}^{[l]^d}$ iff every $k-1$ dimensional face containing the corner $(1, \ldots, 1)$ is empty. That is, if $x = (x_1, \ldots, x_d)$ satisfies*

$$\#\{1 \leq \alpha \leq d \,:\, x_\alpha = 1\} \geq d - k + 1 \tag{5.2.6}$$

then $\eta(x) = 0$. In particular, being framed does not depend on permutations of $\vec{e}_1, \ldots, \vec{e}_d$, but only on their signs.

PROOF. By induction. For $k = 1$ the box is always framed iff the site $(1, \dots, 1)$ is empty, which is exactly the condition given in equation (5.2.6). For $d = k - 1$ equation (5.2.6) is satisfied for all x, therefore the box must be entirely empty. For $d \geq k > 1$ equation (5.2.6) is satisfied iff:

(1) $x_1 = 1$ and $\#\{2 \leq \alpha \leq d : x_\alpha = 1\} \geq (d-1) - k + 1$,
(2) $x_1 \neq 1$ and $\#\{2 \leq \alpha \leq d : x_\alpha = 1\} \geq (d-1) - (k-1) + 1$.

The first part describes equation (5.2.6) for the $(d-1, k)$-frameness of $\{1\} \times [l_2] \times \cdots \times [l_d]$, and the second corresponds to the $(d-1, k-1)$-frameness of each of the faces $\{m\} \times [l_2] \times \cdots \times [l_d]$, $m = 2, \dots, d$. □

COROLLARY 5.2.19. *Fix $l_1, \dots, l_d$ and $\eta \in \Omega_\Lambda$. The box $[l_1] \times \cdots \times [l_d]$ is (d,k)-frameable for η if it is connected by legal FAkf moves in the box to a (k,d)-framed configuration.*

It will sometime be useful to use the convention that boxes are always $(0,d)$-frameable.

DEFINITION 5.2.20. Fix $l_1, \dots, l_d$, and consider the box $[l_1] \times \cdots \times [l_d]$. We say that the box is (k,d)-almost good for the configuration η if all $d-1$ dimensional faces contained in it are $(k-1, d-1)$-frameable for η. The box is good for η if it is almost good for all configurations η' that differ from η by at most one site. For $k = 1$ all boxes are good.

EXAMPLE 5.2.21. For $k = 2$,$d = 2$ a box is good if every column and every line contain at least two empty sites. For $k = 2$, $d = 3$ a box is good if each 2 dimensional section contains at least two empty sites. For $k = 3$, $d = 3$ a box is good if each 2 dimensional section contains at least two empty sites in each row and in each column, and at least two non-intersecting rows or columns are empty.

We continue with more definitions that will help us describing T-step move.

OBSERVATION 5.2.22. Let M be a T-step move taking place in V. Then the loss of M is at most the number of sites in V.

DEFINITION 5.2.23. Fix a T-step move M taking place in $V \subseteq \Lambda$, $\eta \in \mathrm{Dom} M$, and a sequence of permutation $(\sigma_0, \dots, \sigma_T)$ on V, such that $\sigma_{t-1}^{-1}\sigma_t$ is either the identity or a nearest neighbors transposition for all $t \in [T]$. We say that $(\sigma_0, \dots, \sigma_T)$ is compatible with $M\eta$ if for all $t \leq T$ and $x \in \Lambda$

$$M_t \eta\,(x) = \eta\,(\sigma_t x)\,.$$

We say that $M\eta$ and the permutation σ are compatible if there exists such a sequence with $\sigma_T = \sigma$. If the sequence $(\sigma_0, \dots, \sigma_T)$ does not depend on η we say that $M\eta$ and σ are deterministically compatible.

OBSERVATION 5.2.24. Let M be a T-step Λ, and assume that it is deterministically compatible with a permutation σ. Then $\mathrm{Loss} M = 0$.

OBSERVATION 5.2.25. Fix a T-step move M taking place in $V \subseteq \Lambda$. Let σ be a random permutation compatible with M, i.e., for all $\eta \in \text{Dom}M$ the permutation $\sigma(\eta)$ is compatible with $M\eta$. Then there exists a T' step move M' with the same domain that is also compatible with σ, and $T' = |V|!$. If σ is deterministic and M is deterministically compatible with σ, than M' will also be deterministically compatible with σ.

PROOF. There could only be $|V|!$ unique permutations on V, so any sequence $(\sigma_0, \ldots, \sigma_T)$ of length $T \geq T'$ contains two equal permutations $\sigma_t = \sigma_s$ for $t < s$. It could then be shortened by removing the entries $t+1, \ldots, s$. Continue this procedure until $T \leq T'$ and pad it to obtain a T' time move. □

DEFINITION 5.2.26. Let M^1 be a T_1-step move and M_2 a T_2-step move. Assume that for all $\eta \in \text{Dom}M^1$, the configuration $M^1_{T_1}\eta$ is in $\text{Dom}M^2$. Then the composition $M^2 \circ M^1$ will be a $(T_1 + T_2)$-step move with $\text{Dom}M^2 \circ M^1 = \text{Dom}M^1$, such that

$$\left(M^2 \circ M^1\right)_t \eta = \begin{cases} M^1_t \eta & t \leq T_1 \\ M^2_{t-T_1} M_{T_1} \eta & T_1 < t \leq T_2 \end{cases}.$$

OBSERVATION 5.2.27. Fix M^1 a T_1-step move and M^2 a T_2-step move. Then the loss of their composition, when it is defined, is

$$\text{Loss}M^2 \circ M^1 = \max\left\{\text{Loss}M^1, \text{Loss}_{T_1} M^1 + \text{Loss}M^2\right\}.$$

Another operation we on moves is choosing one out of a set of moves for different values of η.

DEFINITION 5.2.28. Let $M^1, \ldots, M^n$ be a sequence of T-step move. Fix a set $\mathcal{M} \subset \Omega_\Lambda$, and a choice function h form $\mathcal{M}$ to $[n]$. Assume that for every $\eta \in \mathcal{M}$

$$\eta \in \text{Dom}M^{h(\eta)}.$$

Then we may define a T-step move M with domain $\mathcal{M}$ such that

$$M_t \eta = M^{h(\eta)}_t \eta.$$

OBSERVATION 5.2.29. Consider a sequence $M^1, \ldots, M^n$ and a choice function h as in the above definition, and let M be the corresponding move. Then

$$\text{Loss}M \leq \log_2 n + \sup_{i \in [n]} \text{Loss}M^i.$$

We are now ready to construct some basic T-step moves, by recursion on d and k, and using previously defined moves on smaller d and k. Unless written otherwise, moves are assumed to be defined for the KAkf in d dimensions.

PROPOSITION 5.2.30. *Fix $x_0 \in \Lambda$, $l_2, \ldots, l_d$. Let $F_1 = x_0 + \{1\} \times [l_2] \times \cdots \times [l_d]$ and $F_2 = x_0 + \{2\} \times [l_2] \times \cdots \times [l_d]$. Then there exists a T-step move M taking place in $F_1 \cup F_2$ satisfying the following properties:*

(1) T is polynomial in $l_2 \dots l_d$,
(2) $DomM = \{\eta \,|\, F_1$ is $(d-1,k)$-framed and F_2 is $(d-1,k-1)$-framed$\}$ (framed refers to the directions $\vec{e}_2, \dots, \vec{e}_d$),
(3) $LossM = 0$,
(4) In the final state the two faces F_1 and F_2 are exchanged – for all $\eta \in DomM$

$$M_T\eta\,(x) = \begin{cases} \eta(x+\vec{e}_\alpha) & x \in F_1 \\ \eta(x-\vec{e}_\alpha) & x \in F_2 \end{cases}.$$

Moreover, M is deterministically compatible with the permutation that exchanges F_1 and F_2.

By isometry this could also be defined for all other orientations. In the following we will refer to this move as the (k,d) framed exchange move of the faces F_1 and F_2. We denote T by $T^{k,d}_{fEx}(l_1, \dots, l_d)$.

Proposition 5.2.31. *Fix $x_0 \in \Lambda$, $l_1, \dots, l_d$ and $x \in [l_1]\times\cdots\times[l_d]$. Let $B = x_0+[l_1]\times\cdots\times[l_d]$. Then there exists a $T^{k,d}$-step move M taking place in B satisfying the following properties:*

(1) T is polynomial in $l_2 \dots l_d$,
(2) $DomM = \{\eta \,|\, B$ is (k,d)-framed in the directions $\vec{e}_1, \dots, \vec{e}_d\}$,
(3) $LossM = 0$,
(4) M is deterministically compatible with a permutation σ that satisfies the following properties:
  (a) σ sends the lower dimensional faces containing the site $(1,\dots,1)$ to the faces containing the site x. More precisely, fix $y = x_0 + (y_1, \dots, y_d)$, and let $y' = x - (1,\dots,1) + (y'_1, \dots, y'_d)$. Then $y_\alpha = 1$ implies $y'_\alpha = 1$ for all $\alpha = 1, \dots, d$. In particular, in the final state all $k-1$ dimensional face containing the site x are empty.
  (b) σ fixes the sites in $x_0 + x + \mathbb{N}^d$, i.e., the sites that are strictly greater than $x_0 + x$ coordinate by coordinate.

By isometry this could also be defined for all other orientations. In the following we will refer to this move as the (k,d) sliding move of the frame of B into x. We denote T by $T^{k,d}_{Sld}(l_1, \dots, l_d)$.

Proposition 5.2.32. *Fix $x_0 \in \Lambda$, and let M_1 be a T_1-step move taking place in $x_0 + \{2\} \times [l_2] \times \cdots \times [l_d]$ for the KA$(k-1)f$ dynamics in dimension $d-1$. Then there exits a T-step move M taking place in $x_0 + \{1,2\} \times [l_2] \times \cdots \times [l_d]$ for the KAkf dynamics satisfying the following properties:*

(1) $T = 2T^{k,d-1}_{Sld} T_1$,
(2) $DomM = DomM_1 \cap \{\eta \,:\, x_0 + \{1\} \times [l_2] \times \cdots \times [l_d]$ is $(k,d-1)$-framed$\}$,
(3) $LossM = LossM_1$,

(4) If M_1 is compatible with a permutation σ than M is also compatible with σ. Moreover, if M_1 is deterministically compatible with σ than M will also be deterministically compatible with σ.

By isometry this could also be defined for all other orientations. In the following we will refer to this move as the application of M_1 on the face $x_0 + \{2\} \times [l_2] \times \cdots \times [l_d]$ with the help of the face $x_0 + \{1\} \times [l_2] \times \cdots \times [l_d]$.

From now on, in order to simplify notation, we only consider $l_1 = \cdots = l_d = l$.

PROPOSITION 5.2.33. *Fix $x_0 \in \Lambda$ and $\zeta_1, \ldots, \zeta_d \in \{+, -\}$. Then there exists a T-step move M taking place in $x_0 + [l]^d$ satisfying the following properties:*

(1) For $k = 1$ the time $T \leq dl$, and for $k \geq 2$ it is bounded by $T \leq 2^{l^d}$,
(2) $DomM = \left\{\eta \,:\, \text{the box } x_0 + [l]^d \text{ is } (d,k)\text{-frameable}\right\}$,
(3) For $k = 1$ $LossM \leq d \log l$ and for $k \geq 2$ $LossM \leq l^d$.
(4) In the final state the box $x_0 + [l]^d$ is framed in the directions $\zeta_1 e_1, \ldots, \zeta_d e_d$.

In the following we will refer to this move as the framing move of the box $x_0 + [l]^d$ in the directions $\zeta_1 e_1, \ldots, \zeta_d e_d$. We denote T by $T_{Fr}^{k,d}(l_1, \ldots, l_d)$.

In some cases we will need to frame boxes will smaller loss of information.

PROPOSITION 5.2.34. *Fix $x_0 \in \Lambda$ and $\zeta_1, \ldots, \zeta_d \in \{+, -\}$. Then there exists a T-step move M taking place in $x_0 + [l]^d$ satisfying the following properties:*

(1) For $k = 1$ and $k = 2$ the time T is bounded by a polynomial in l, and for $k \geq 3$ $T \leq 2^{l^d}$,
(2) $DomM = \left\{\eta \,|\, x_0 + \{1\} \times [l]^{d-1} \text{ is } (d-1,k)\text{-framed and } x_0 + [l]^d \text{ is } (d,k) \text{ almost good}\right\}$,
(3) For $k = 1$ $LossM \leq d \log l$, for $k = 2$ $LossM \leq d^2 l \log l$, and for $k \geq 3$ $LossM \leq l^d$,
(4) In the final state the box $x_0 + [l]^d$ is framed in the directions $\zeta_1 e_1, \ldots, \zeta_d e_d$.

In the following we will refer to this move as the information saving framing move of the box $x_0 + [l]^d$ in the directions $\zeta_1 e_1, \ldots, \zeta_d e_d$. We denote T by $T_{Fr}^{k,d}(l_1, \ldots, l_d)$.

PROPOSITION 5.2.35. *Fix $x_0 \in \Lambda$ and a permutation σ of the sites in $x_0 + [l]^d$. Assume that σ fixes the frame defined as the set of sites satisfying 5.2.6. Then there exists a T-step move M taking place in $x_0 + [l]^d$ satisfying the following properties:*

(1) T is polynomial in l,
(2) $DomM = \left\{\eta \,|\, \text{the box } x_0 + [l]^d \text{ is framed in the directions } e_1, \ldots, e_d\right\}$,
(3) $LossM = 0$,
(4) The permutation σ is deterministically compatible with M.

By isometry this could also be defined for boxes framed in all other directions. In the following we will refer to this move as the permutation move that applies σ to $x_0 + \{2, \ldots, l\}^d$. We denote T by $T_{Per}^{k,d}$.

PROPOSITION 5.2.36. *Fix $x_0 \in \Lambda$ and $e, e' \in \{\pm e_1, \dots, \pm e_d\}$. Then there exists a T-step move M taking place in $x_0 + [l]^d$ satisfying the following properties:*

1. *T is polynomial in l,*
2. *$DomM = \left\{\eta \,|\, \text{the box } x_0 + [l]^d \text{ is framed in the directions } \vec{e}_1, \dots, \vec{e}_d\right\}$,*
3. *$LossM = 0$*
4. *It the final state the box $x_0 + [l]^d$ is reflected according to e, e':*
$$M_T\eta\,(x) = \eta\,(R_{e,e'}x)\,.$$
Moreover, the move is deterministically compatible with the permutation that sends $x_0 + x$ to $x_0 + R_{e,e'}x$

In the following we will refer to this move as the (k,d) framed reflection move of $x_0 + [l]^d$ in the directions e, e'. We denote T by $T^{k,d}_{fRef}$.

PROPOSITION 5.2.37. *Fix $x_0 \in \Lambda$ and $e, e' \in \{\pm e_1, \dots, \pm e_d\}$. Then there exists a T-step move M taking place in $x_0 + [l]^d$ satisfying the following properties:*

1. *For $k = 1$ and $k = 2$ the time T is bounded by a polynomial in l, and for $k \geq 3$ $T \leq 2^{l^d} \times$ polynomial in l,*
2. *$DomM = \left\{\eta \,|\, x_0 + [l]^d \text{ is } (k,d) \text{ almost good and } x_0 + \{1\} \times [l]^{d-1} \text{ is } (d-1,k)\text{-framed}\right\}$,*
3. *$Loss_T M = 0$,*
4. *For $k = 1$ $LossM \leq d \log l$, for $k = 2$ $LossM \leq d^2 l \log l$, and for $k \geq 3$ $LossM \leq l^d$,*
5. *It the final state the box $x_0 + [l]^d$ is reflected according to e, e':*
$$M_T\eta\,(x) = \eta\,(R_{e,e'}x)\,.$$
That is, the move is compatible with the permutation that sends $x_0 + x$ to $x_0 + R_{e,e'}x$

In the following we will refer to this move as the (k,d) reflection move of $x_0 + [l]^d$ in the directions e, e'. We denote T by $T^{k,d}_{Ref}$.

PROPOSITION 5.2.38. *Fix $x_0 \in \Lambda$, $e \in \{\pm e_1, \dots, \pm e_d\}$ and $x^* = \left(x^*_1, \dots, x^*_{d-1}\right) \in [l]^{d-1}$. Let α be such that $e \in \{\pm e_\alpha\}$ and $x^*_e = x_0 + \left(x^*_1, \dots, x^*_{\alpha-1}, 0, x^*_\alpha, \dots, x^*_{d-1}\right)$. Then there exists a T-step move M taking place in $x_0 + [l]^{\alpha-1} \times \{-1, 0, 1\} \times [l]^{d-\alpha}$ satisfying the following properties:*

1. *For $k = 2$ and $k = 3$ the time T is bounded by a polynomial in l, and for $k \geq 4$ $T \leq 2^{l^d} \times$ polynomial in l,*
2. *For a fixed configuration η, let η^* be the configuration that equals η for $x \notin \{x^*_e + e, x^*_e - e\}$ and 1 for $x \in \{x^*_e + e, x^*_e - e\}$. With this notation,*
$$\begin{aligned} DomM \;=\; & \Big\{\eta \,|\, \text{the face } x_0 + [l]^{\alpha-1} \times \{1\} \times [l]^{d-\alpha} \text{ and the face} \\ & x_0 + [l]^{\alpha-1} \times \{-1\} \times [l]^{d-\alpha} \text{ are } (k-1, d-1) \text{ frameable for } \eta^*, \\ & \text{and the face } x_0 + [l]^{\alpha-1} \times \{0\} \times [l]^{d-\alpha} \text{ is } (k, d-1) \text{ framed}\Big\}, \end{aligned}$$
3. *For $k = 2$ $LossM \leq 2d \log l$, for $k = 3$ $LossM \leq 2d^2 l \log l$, and for $k \geq 4$ $LossM \leq 2l^d$.*

(4) It the final state

$$M_T\eta\,(x)=\begin{cases}\eta\,(x_e^*+e) & x=x_e^*-e\\ \eta\,(x_e^*-e) & x=x_e^*+e\ ,\\ \eta\,(x) & \textit{otherwise}\end{cases}$$

i.e., $M\eta$ is compatible with the permutation that exchanges x_e^*+e and x_e^*-e.

In the following we will refer to this move as the jump move of the site x_e^*-e to x_e^*+e. We denote T by $T_{Jmp}^{k,d}$.

For the construction of the next moves we will use the notion of geometric paths, which are the same as the paths of the previous sections but with some extra information.

Definition 5.2.39. A geometric path of length n is a sequence $i_1,\dots,i_n\in\left(\mathbb{Z}_l^d\right)^n$, together with two directions $e_{\text{in}}^{(1)},e_{\text{out}}^{(n)}\in\{\pm e_1,\dots,\pm e_d\}$, such that $i_{\tau+1}-i_\tau\in\{\pm e_1,\dots,\pm e_d\}$ for all $\tau<n$. We will denote, at each step τ, $e_{\text{in}}^{(\tau)}=i_\tau-i_{\tau-1}$ and $e_{\text{out}}^{(\tau)}=i_\tau-i_{\tau+1}$. We say that the geometric path is good if for a configuration η of the box B_{i_τ} is good for all τ, and super-good if it is good and the face $li_1+R_{e_1,e_{\text{in}}^{(1)}}\left(\{1\}\times[l]^{d-1}\right)$ is $(k,d-1)$-frameable for all configurations that differ from η at at most one site.

We will also need to keep track of the time a move spends at some fixed box

Definition 5.2.40. Consider a T-step move M and some set $V\subset\mathbb{Z}^d$. Then the time M spends in V for a configuration η is

$$\mathcal{T}_{M,\eta}^V=\{t\in[T]\,:\,\exists x\in B_j\text{ such that }M_t\eta\,(x)\neq M_{t+1}\eta\,(x)\}\,.$$

We denote $\mathcal{T}_M^V=\cup_{\eta\in\text{Dom}M}\mathcal{T}_{M,\eta}^V$. When $V=B_i$ for some $i\in\mathbb{Z}_l^d$ we will use the notation $\mathcal{T}_{M,\eta}^{(i)}$ and $\mathcal{T}_M^{(i)}$.

Proposition 5.2.41. *Fix a geometric path $i_1,\dots,i_n$ with directions $e_{in}^{(1)},e_{out}^{(n)}$, and $x^*\in[l]^{d-1}$. Let $x_{e_{in}^{(1)}}^*=li_1+R_{e_1,e_{in}^{(1)}}(2,x^*)$ (in analogy with Proposition 5.2.38) and $x_{e_{out}^{(n)}}^*=li_n+R_{e_1,e_{out}^{(n)}}(2,x^*)$. Then there exists a T-step move M taking place in $\bigcup_{\tau=1}^n B_{i_\tau}$ satisfying the following properties:*

(1) $T=n\times$ polynom in l for $k=1$ or $k=2$, and $T\leq 2^{l^d}\times$ polynomial in l for $k\geq 3$,
(2) $DomM=\{\eta\,:\,\text{the path is super-good}\}$,
(3) For $k=1$ $LossM\leq 2d\log l$, for $k=2$ $LossM\leq 2d^2l\log l$, and for $k\geq 3$ $LossM\leq 2l^d$,
(4) $\left|\mathcal{T}_M^{(j)}\right|$ is bounded by $\frac{2T}{n}$, uniformly for all $j\in\mathbb{Z}_l^d$,
(5) The path $i_n,\dots,i_1$ with directions $e_{out}^{(n)},e_{in}^{(1)}$ is super-good for the configuration $M_T\eta$.
(6) There exists a permutation σ, independent of η, such that
(a) σ is compatible with $M\eta$ for all $\eta\in DomM$,
(b) $\sigma x_{e_{in}^{(1)}}^*=x_{e_{out}^{(n)}}^*$.

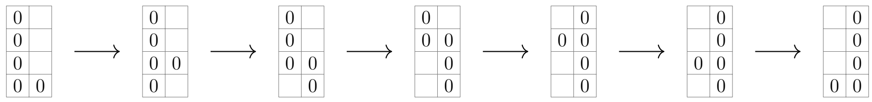

FIGURE 5.2.4. The framed exchange move for $k = d = 2$

PROPOSITION 5.2.42. *Fix $i \in \mathbb{Z}_l^d$ and $x^* = (x_1^*, x_2^*, \ldots, x_d^*) \in [l]^d$. Then there exists a T-step move M taking place in $B_i \cup Q_i$ satisfying the following properties:*

(1) $T =$ polynom in l,

(2) For every box $j \in Q_i$, we write $j = \sum_{\alpha \in d} c_\alpha \vec{e}_\alpha$ for $c_\alpha \in \{0, 1\}$. Let $c'_\alpha = \begin{cases} 1 & c_\alpha = 1 \\ -1 & c_\alpha = 0 \end{cases}$.

Then

$$\text{Dom}M = \{\eta : \text{For ever } j \in Q_i \text{ the box } B_j \text{ is } (k,d)\text{-framed in the directions } c'_1\vec{e}_1, \ldots, c'_d\vec{e}_d\},$$

(3) $\text{Loss}M = 0$,

(4) M is deterministically compatible with the permutation that swaps the sites $li + x^*$ and $l(i + e_1) + (2, x_2^*, \ldots, x_d^*)$.

In the following we will refer to this move as the swap move of $li + x^*$ and $l(i + e_1) + (2, x_2^*, \ldots, x_d^*)$.

We are finally ready to show how to flip a single site.

PROPOSITION 5.2.43. *Fix a geometric path $i_1, \ldots, i_n, i_{n+1}$ with directions $e_{in}^{(1)}, e_{out}^{(n)}$ such that $i_1 - e_{in}^{(1)} \notin \Lambda_l$. Fix $x^* \in [l]^d$. Set $i = i_{n+1}$. Then there exists a T-step move M taking place in $\bigcup_{\tau=1}^{n} B_{i_\tau} \cup Q_i$ satisfying the following properties:*

(1) $T = n \times$ polynom in l for $k = 1$ or $k = 2$, and $T = 2^{l^d} \times$ polynomial in l for $k \geq 3$,

(2) $\text{Dom}M = \{\eta : \text{the path } i_1, \ldots, i_n \text{ is good and the boxes } Q_i \text{ are good}\}$,

(3) For $k = 1$ $\text{Loss}M \leq 10d \log l$, for $k = 2$ $\text{Loss}M \leq 10d^2 l \log l$, and for $k \geq 3$ $\text{Loss}M \leq 10l^d$,

(4) $E(M) \leq C_{k,d} l^{k-1}$, where $C_{k,d}$ is a positive constant that may depend on k and d,

(5) $\left|\mathcal{T}_M^{(j)}\right|$ is bounded by $\frac{2T}{n}$, uniformly for all $j \in \mathbb{Z}_l^d$,

(6) $M_T \eta = \eta^{li+x^*}$, i.e., the site $li + x^*$ is flipped.

The move given in this proposition is the one we take for Proposition 5.2.14.

We can now construct these moves.

5.2.4.1. *The framed exchange move (Proposition 5.2.30).* Assume WLOG $x_0 = 0$, $l_2 \leq \cdots \leq l_d$, and take $\eta \in \text{Dom}M$. Note that the definition of $\text{Dom}M$ is equivalent to the requirement that the box $[2] \times [l_2] \times \cdots \times [l_d]$ is (k,d)-framed in the directions $\vec{e}_1, \ldots, \vec{e}_d$.

For $k = 1$, the KA-1f dynamics is the unconstrained one, so we may simply exchange the two frames site by site.

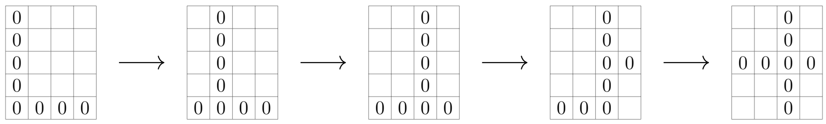

FIGURE 5.2.5. The frame sliding move for $k = d = 2$

For $l_2 = 2$, by Proposition 5.2.18, the box $[2]\times\{1\}\times[l_3]\times\cdots\times[l_d]$ is $(k, d-1)$ framed – being framed does not depend on the order in which we take the directions, therefore $[2]\times[l_2]\times\cdots\times[l_d]$ is (k,d)-framed also in the directions $\vec{e}_2, \vec{e}_1, \vec{e}_3, \ldots, \vec{e}_d$. We can thus use Proposition 5.2.32 in order to apply $(d-1, k-1)$ moves to $[2]\times\{2\}\times[l_3]\times\cdots\times[l_d]$. We also know that $[2]\times\{2\}\times[l_3]\times\cdots\times[l_d]$ is $(k-1, d-1)$ framed, so η is in the domain of the $(k-1, d-1)$ frame exchange move that exchanges $\{1\}\times\{2\}\times[l_3]\times\cdots\times[l_d]$ and $\{2\}\times\{2\}\times[l_3]\times\cdots\times[l_d]$. Proposition 5.2.32 thus allows us to exchange these two faces. In order to finish we need to exchange $\{1\}\times\{1\}\times[l_3]\times\cdots\times[l_d]$ and $\{2\}\times\{1\}\times[l_3]\times\cdots\times[l_d]$. This could be done using the $(k, d-1)$ framed exchange move – $[2]\times\{1\}\times[l_3]\times\cdots\times[l_d]$ is $(k, d-1)$ framed, so η is indeed in its domain.

If $l_2 > 2$, we start by exchanging the faces $[2]\times\{1\}\times[l_3]\times\cdots\times[l_d]$ and $[2]\times\{2\}\times[l_3]\times\cdots\times[l_d]$. This could be done since these two faces are in the domain of the (k,d)-frame exchange move, with faces whose smallest side is 2. For this new configuration, will will first consider the sites of $[2]\times\{1\}\times[l_3]\times\cdots\times[l_d]$, and then the sites of $[2]\times\{2,\ldots,l_2\}\times[l_3]\times\cdots\times[l_d]$.

The face $[2]\times\{2\}\times[l_3]\times\cdots\times[l_d]$ is now $(k, d-1)$ framed, so we will use Proposition 5.2.32 in order to apply $(d-1, k-1)$ moves to $[2]\times\{1\}\times[l_3]\times\cdots\times[l_d]$. The face $[2]\times\{1\}\times[l_3]\times\cdots\times[l_d]$ is $(k-1, d-1)$-framed in the directions $\vec{e}_1, e_3, \ldots, \vec{e}_d$. That is, $\{1\}\times\{1\}\times[l_3]\times\cdots\times[l_d]$ is $(k-1, d-2)$-framed and $\{2\}\times\{1\}\times[l_3]\times\cdots\times[l_d]$ is $(k-2, d-2)$-framed. This allows us to exchange these two faces using the $(k-1, d-2)$ framed exchange move.

The box $[2]\times\{2,\ldots,l_2\}\times[l_3]\times\cdots\times[l_d]$ is (k,d)-framed in the directions $\vec{e}_1, \ldots, \vec{e}_d$, so we apply (by induction) the (k,d) framed exchange move on it with a smaller value of l_2.

Finally, we exchange back the two faces $[2]\times\{1\}\times[l_3]\times\cdots\times[l_d]$ and $[2]\times\{2\}\times[l_3]\times\cdots\times[l_d]$, and this conclude the construction of the move. See figure 5.2.4.

5.2.4.2. *The frame sliding move (Proposition 5.2.31).* Assume WLOG $x_0 = 0$, and set $x = (x_1, \ldots, x_d)$. Use the framed exchange move x_1 times in order to move the face $\{1\}\times[l_2]\times\cdots\times[l_d]$ to $\{x_1\}\times[l_2]\times\cdots\times[l_d]$. Consider now the box $\{x_1, \ldots, l_1\}\times[l_2]\times\cdots\times[l_d]$. It is framed in the directions $\vec{e}_1, \ldots, \vec{e}_d$, so we can apply the sliding move by induction into the point x. Similarly, the box $[x_1]\times[l_2]\times\cdots\times[l_d]$ is framed in the directions $-\vec{e}_1, \ldots, \vec{e}_d$, and again we can apply the sliding move into the point x. See figure 5.2.5.

5.2.4.3. *Performing $(k-1, d-1)$ moves near a framed face (Proposition 5.2.32).* For each exchange of M of two sites x, y, start by sliding the frame of $x_0 + \{1\}\times[l_2]\times\cdots\times[l_d]$ to a the site $x - \vec{e}_1$. Then both x and y get an extra empty neighbor.

5.2.4.4. *Framing a box (Proposition 5.2.33).* Assume WLOG that $\zeta_1 = \cdots = \zeta_d = +$. Start with $k = 1$, i.e., the unconstrained case. For $x \in x_0 + [l]^d$ we will use a T-step move M^x whose

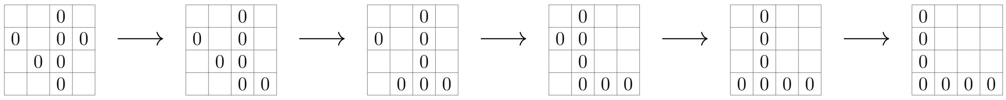

FIGURE 5.2.6. The box framing move for $k = d = 2$

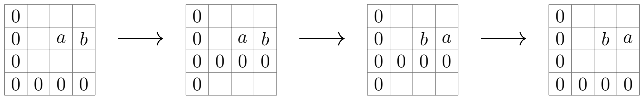

FIGURE 5.2.7. The permutation move for $k = d = 2$

domain is $\{\eta : x \text{ is empty}\}$, deterministically compatible with a permutation σ that satisfies $\sigma x = x_0 + (1, \dots, 1)$. Since the dynamics is not constrained, we can construct such a move with $T = dl$. We then use the choice function defined for $\eta \in \text{Dom}M$, that assigns to η the move M^x, for $x \in x_0 + [l_1] \times \cdots \times [l_d]$ that is empty for η (when ambiguous we take the minimal one in some arbitrary order). This function is well defined since $\eta \in \text{Dom}M$ means (for $k = 1$) that at least one of the sites in $x_0 + [l]^d$ is empty. We can then define M as in Definition 5.2.28, and Observation 5.2.29 completes the proof for this case.

For $k > 1$, by definition of frameability there exists a move that frames the box. The bounds on the time and on the loss come from the fact that there are at most 2^{l^d} configurations in the box.

See figure 5.2.6.

5.2.4.5. *Information saving framing.* Assume WLOG that $x_0 = 0$ and $\zeta_1 = \cdots = \zeta_d = +$. Note that for $\eta \in \text{Dom}M$ the box x_0 is frameable, so it suffices to consider $k = 2$.

The face $\{1\} \times [l]^{d-1}$ is framed, therefore by Proposition 5.2.32 we can apply $(d-1, 1)$ moves to $\{2\} \times [l]^{d-1}$. This face is $(d-1, 1)$-frameable, so we may apply the framing move in order to frame it. Then we use the framed exchange move and exchange $\{1\} \times [l]^{d-1}$ with $\{2\} \times [l]^{d-1}$. In the same way we frame $\{3\} \times [l]^{d-1}$ and exchange it with $\{2\} \times [l]^{d-1}$, until framing $\{l\} \times [l]^{d-1}$. We can then exchange back $\{l-1\} \times [l]^{d-1}$ with $\{l-2\} \times [l]^{d-1}$ and so on until it is back to $\{1\} \times [l]^{d-1}$. By Observation 5.2.27 and the fact that $(d-1, 1)$ framing has loss $d \log l$ the result follows.

5.2.4.6. *The permutation move (Proposition 5.2.35).* Assume first that σ is a transposition that exchanges two neighboring sites, without loss of generality $x = (x_1, \dots, x_d)$ and $y = x + \vec{e}_d$. If $k = 1$ we can simply exchange the two sites. Consider now $k \geq 2$. We know that x is not on the frame of $[l]^d$, so at least k of its coordinates are different than 1. We can assume that one of these coordinates is the first one, i.e., $x_1 > 1$. We will then slide the frame of $[l]^d$ to the point $(x_1 - 1, 1, \dots, 1)$, without changing the sites of $[x_1, l] \times [l]^d$. After this move, the face $\{x_1 - 1\} \times [l]^{d-1}$ is $(k, d-1)$ framed, so we may apply $(k-1, d-1)$ moves to the face $\{x_1\} \times [l]^{d-1}$. Note that $\{x_1\} \times [l]^{d-1}$ is $(k-1, d-1)$ framed and both x and y are contained in it. Moreover, x and y are not on the frame of $\{x_1\} \times [l]^{d-1}$, so we can use the lower dimensional permutation move to exchange them. Then rewind the sliding move finishes the construction.

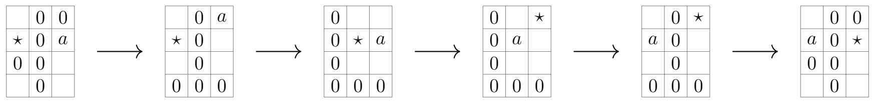

FIGURE 5.2.8. The jump move for $k = d = 2$

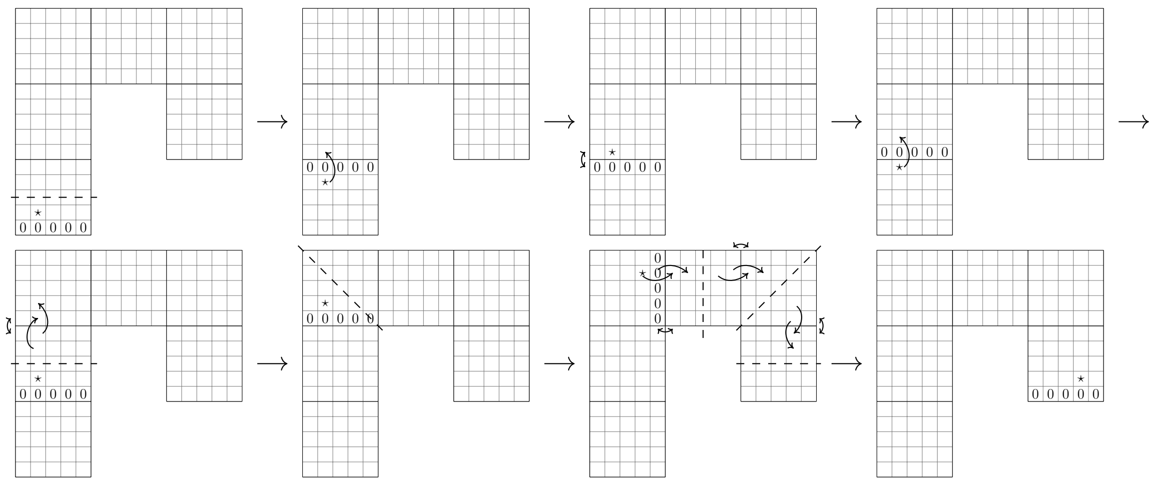

FIGURE 5.2.9. Propagating a site along a path for $k = d = 2$

Finally, we note that every transposition can be written as a product of at most $2dl$ nearest-neighbor transposition, and every permutation can be written as the sum of at most $l^d \log l^d$ permutations. This concludes the proof. See figure 5.2.7.

5.2.4.7. *The framed reflection move (Proposition 5.2.36).* Assume without loss of generality $x_0 = 0$. First, slide the frame into the point $R_{e,e'}(1, \ldots, 1)$. The frame of the resulting configuration is indeed the reflected frame, and for the other site we can use Proposition 5.2.35.

5.2.4.8. *The reflection move (Proposition 5.2.37).* In order to reflect a box, we start by framing it in the direction $\vec{e}_1, \ldots, \vec{e}_d$. We then apply a framed reflection, and finally unwrap the framing move that we have applied in the beginning.

5.2.4.9. *The jump move (Proposition 5.2.38).* Start by framing the face $x_0 + [l]^{\alpha-1} \times \{0\} \times [l]^{d-\alpha}$. Then, by Proposition 5.2.32 we can also frame the faces $x_0 + [l]^{\alpha-1} \times \{\pm 1\} \times [l]^{d-\alpha}$ for η^*. When framing these faces, $x_e^* - \vec{e}_1$ and $x_e^* + \vec{e}_1$ could change position, so we denote by $y - \vec{e}_1$ and $y' + \vec{e}_1$ their new positions. We can now exchange $x_0 + [l]^{\alpha-1} \times \{0\} \times [l]^{d-\alpha}$ and $x_0 + [l]^{\alpha-1} \times \{-1\} \times [l]^{d-\alpha}$ using the framed exchange, permute x'_e with $y' + \vec{e}_1$ with the permutation move, and exchange again $x_0 + [l]^{\alpha-1} \times \{0\} \times [l]^{d-\alpha}$ and $x_0 + [l]^{\alpha-1} \times \{-1\} \times [l]^{d-\alpha}$. All that is left is to wind back the framing moves and we are done. See figure 5.2.8.

5.2.4.10. *Propagating a site along a path (Proposition 5.2.41).* We construct this move by induction over n. For $n = 1$ the reflection move of the box B_{i_1} in the directions $e_{\text{in}}, e_{\text{out}}$ will give the result. For $n > 1$, propagate the site along the first $n - 1$ boxes. We can then apply the jump move swapping $x^*_{e^{(n-1)}_{\text{out}}}$ with $x^*_{e^{(n)}_{\text{in}}}$. This gives the required move. See figure 5.2.9.

5.2.4.11. *The swap move (Proposition 5.2.42).* Note that for $\eta \in \text{Dom}M$ the box $i + [l+1]^d$ is framed in the directions $-\vec{e}_1, \ldots, -\vec{e}_d$. We can thus apply the permutation move in order to move the site $li+x^*$ to $li+(l,x_2^*,\dots,x_d^*)$. Then the jump move allows us to swap $li+(l,x_2^*,\dots,x_d^*)$ with $li+(l+2,x_2^*,\dots,x_d^*)=l\,(i+e_1)+(2,x_2^*,\dots,x_d^*)$, and then applying the inverse permutation finishes the construction.

5.2.4.12. *Flipping a site.* The first step is to frame the boxes of $Q_i\setminus\{i+\vec{e}_1\}$. By creating $O\left(l^{k-1}\right)$ zeros on the boundary, we can make the box i_1 frameable. Then we apply Proposition 5.2.41 $d-1$ times (with arbitrary x^*), and framing $Q_i\setminus\{i+\vec{e}_1\}$ in the directions required by Proposition 5.2.41. We can do that since by adding sites of Q_i to the path $i_1,\dots,i_n$ we can obtain a super-good path that reaches each of the boxes boxes in Q_i. We then use the boundary condition and set the site $li_1+R_{e_1,e_{\mathrm{in}}^{(1)}}(2,x_2^*,\dots,x_d^*)$ to have the occupation value $1-\eta\,(li+x^*)$. Then propagate the site along the path using Proposition 5.2.41 and swap it with $li+x^*$ using Proposition 5.2.42. We now roll back everything, and the proof is finished.

5.2.5. From the long range Poincaré inequality to the Kob-Andersen dynamics. In this section we bound from above the Dirichlet form with the long range constraints appearing in the r.h.s. of (5.2.5) with that of the KA model in Λ (5.2.2). Given $i\in\Lambda_l$ our aim is to bound the quantity $\mu(\hat{c}_i\mathrm{Var}_{B_i}(f)$ appearing in the r.h.s. of (5.2.5) using the T-step moves that have been constructed in the previous section. In order to do that, it is convenient to first condition on the environment of the coarse-grained variables $\{\mathbb{1}_{\{B_j\text{ is good}\}}\}_{\substack{j\in\Lambda_l\\ j\neq i}}$. The main advantage of the above conditioning is that the good family for the vertex $i+\vec{e}_1$, whose existence is guaranteed by the long range constraint $\hat{c}_i$, become deterministic. We will thus work first in a fixed realization of the coarse-grained variables satisfying $\hat{c}_i=1$ and only at the end we will take an average and we will sum over i. The main technical step of the above program is as follows.

Given $i\in\Lambda_l$ let γ be an up-right focused path γ of length $2N$ starting at $i+\vec{e}_1$ and let $G_{i,\gamma}$ be the event that γ is good and all $j\in Q_i\cap\Lambda_l$ are good. Let also $V_{i,\gamma}:=B_i\cup_{j\in\gamma\cup Q_i}B_j$ and let $\mathcal{F}_i$ be the σ-algebra generated by the random variables $\{\mathbb{1}_{\{B_j\text{ is good}\}}\}_{\substack{j\in\Lambda_l\\ j\neq i}}$. Notice that $G_{i,\gamma}$ is measurable w.r.t. $\mathcal{F}_i$. Finally write

$$\mathcal{D}_{i,\gamma}(f\,|\,\mathcal{F}_i):=\sum_{\substack{x,y\in V_{i,\gamma}\cap\Lambda\\ \|x-y\|_1=1}}\mu\big(c_{xy}\left(\nabla_{xy}f\right)^2\,|\,\mathcal{F}_i\big)+\sum_{x\in V_{i,\gamma}\cap\partial\Lambda}\mu\big(\mathrm{Var}_x(f)\,|\,\mathcal{F}_i\big).$$

Clearly the average w.r.t. μ of $\mathcal{D}_{i,\gamma}(f\,|\,\mathcal{F}_i)$ represents the contribution coming from the set $V_{i,\gamma}\cap\Lambda$ to the total Dirichlet form $\mathcal{D}(f)$.

LEMMA 5.2.44. *On the event $G_{i,\gamma}$*

$$\mu\big(\mathit{Var}_{B_i}(f)\,|\,\mathcal{F}_i\big)\le O(N)e^{O(l^{d-1}(|\log(q)|+\log(l)))}\,\mathcal{D}_{i,\gamma}(f\,|\,\mathcal{F}_i)\quad\forall f:\mathcal{E}_\Lambda\mapsto\mathbb{R}.$$

for $k=2$, and

$$\mu\big(\mathit{Var}_{B_i}(f)\,|\,\mathcal{F}_i\big)\le O(N)e^{O(l^{d}(|\log(q)|+\log(l)))}\,\mathcal{D}_{i,\gamma}(f\,|\,\mathcal{F}_i)\quad\forall f:\mathcal{E}_\Lambda\mapsto\mathbb{R}.$$

when $k\geq 2$.

PROOF. Assume $\mathbb{1}_{G_{i,\gamma}} = 1$. Since the marginal of $\mu(\cdot \mid \mathcal{F}_i)$ on $\{0,1\}^{B_i}$ is a product measure we have immediately that

$$\mu\big(\mathrm{Var}_{B_i}(f) \mid \mathcal{F}_i\big) \le \sum_{x \in B_i} \mu\big(\mathrm{Var}_x(f) \mid \mathcal{F}_i\big),$$

and it is sufficient to prove that

$$\max_{x \in B_i} \mu\big(\mathrm{Var}_x(f) \mid \mathcal{F}_i\big) \le O(N) e^{O(l^{d-1}(|\log(q)|+\log(l)))} \mathcal{D}_{i,\gamma}(f \mid \mathcal{F}_i)$$

for $k = 2$, and with l^d in the exponent for $k > 2$. Given $x \in B_i$, Proposition 5.2.14 and the assumption $\mathbb{1}_{G_{i,\gamma}} = 1$ imply that there exists a T-step move M with $\mathrm{Dom(M)} = G_{i,\gamma}$, taking place in $V_{i,\gamma} \cap \Lambda$ and such that for all $\eta \in \mathrm{Dom(M)}$ $M_T\eta$ is the configuration η flipped at x. Notice that M does not change the variables $\{\mathbb{1}_{\{B_j \text{ is good}\}}\}_{\substack{j \in \Lambda_l \\ j \neq i}}$. Hence, on the event $G_{i,\gamma}$,

$$\begin{aligned}
\mathrm{Var}_x(f) &= pq \left(f(\eta) - f(\eta^x)\right)^2 \\
&\le \Big(\sum_{t=0}^{T-1} \left(f(M_t\eta) - f(M_{t+1}\eta)\right) \Big)^2 \\
&\le T \sum_{t=0}^{T-1} \left(f(M_t\eta) - f(M_{t+1}\eta)\right)^2.
\end{aligned} \tag{5.2.7}$$

In order to proceed it is convenient to introduce the following notation.

A pair of configurations $e = (\eta, \eta') \in \mathcal{E}^2$ is called a *KA-edge* if $\eta \neq \eta'$ and η' is obtained from η by applying to η either a legal exchange at some bond b_e of Λ or a spin flip at some site $z_e \in \partial\Lambda$. If b_e or z_e belong to a given $V \subset \Lambda$ we say that *the edge e occurs in V*. Given a KA-edge $e = (\eta, \eta')$ we write $\nabla_e f := f(\eta') - f(\eta)$. Finally the collection of all KA-edges in $\mathcal{E}^2$ is denoted $\mathcal{E}_{\mathrm{KA}}$.

By construction, if $M_{t+1}\eta \neq M_t\eta$ then $e_t := (M_t\eta, M_{t+1}\eta)$ is a KA-edge and the r.h.s. of (5.2.7) can be written as

$$T \sum_{t=0}^{T-1} c_{e_t} \Big(\nabla_{e_t} f\Big)^2,$$

where c_{e_t} is the kinetic constraint associated to the KA-edge e_t. Taking the expectation over η w.r.t. $\mu(\cdot \mid \mathcal{F}_i)$ yields

$$\mu\Big(\mathrm{Var}_x(f) \mid \mathcal{F}_i\Big) \le T \sum_{e \in \mathcal{E}_{\mathrm{KA}}} \sum_{t=0}^{T} \mu\big(c_e \big(\nabla_e f\big)^2 \mathbb{1}_{\{e = (M_t\eta, M_{t+1}\eta)\}} \mid \mathcal{F}_i\big). \tag{5.2.8}$$

Next we use the following chain of observations (recall Proposition 5.2.14 and the relevant definitions therein).

(1) For any KA-edge e and any η such that $e = (M_t\eta, M_{t+1}\eta)$ for some $t \le T$ it holds that (for $q < 1/2$)

$$\mu(\eta) \le q^{-E(M)} \mu(M_t\eta).$$

(2) Since the T-move M takes place in the set $V_{i,\gamma} \cap \Lambda$, in the r.h.s. of (5.2.8) we can replace $\sum_{e \in \mathcal{E}_{\mathrm{KA}}}$ by

$$\sum_{\substack{e \in \mathcal{E}_{\mathrm{KA}} \\ e \text{ occurs in } V_{i,\gamma} \cap \Lambda}} .$$

(3) Given a KA-edge e occurring in some $B_j \subset V_{i,\gamma} \cap \Lambda$,

$$\sum_{\eta \in \Omega} \sum_{t=1}^{T} \mathbb{1}_{\{e=(M_t\eta, M_{t+1}\eta)\}} \leq 2^{\mathrm{Loss(M)}} |\mathcal{T}_M^{(j)}|.$$

Using the above remarks, on the event $G_{i,\gamma}$,

$$\mu\big(\mathrm{Var}_x(f) \,|\, \mathcal{F}_i\big) \leq T\, 2^{\mathrm{Loss(M)}} |\mathcal{T}_{\mathrm{M}}^{(j)}| q^{E(\mathrm{M})} \sum_{\substack{e=(\eta,\eta') \in \mathcal{E}_{\mathrm{KA}} \\ e \text{ occurs in } V_{i,\gamma} \cap \Lambda}} \mu(\eta \,|\, \mathcal{F}_i) c_e(\eta) \big(\nabla_e f\big)^2.$$

This expression, by Proposition 5.2.14, satisfies the required bound. □

We are now ready to state the main result of this section. For simplicity in the sequel we shall write $C(l,q)$ for any positive function such that

$$C(l,q) = e^{O(l^{d-1}(|\log(q)| + \log(l)))}, \quad \text{as } l \uparrow +\infty, q \downarrow 0. \tag{5.2.9}$$

for $k=2$, and with l^d in the exponent for $k \geq 2$. Of course the constant in the $O(\cdot)$ notation may change from line to line.

PROPOSITION 5.2.45. *Let $\mathcal{D}^l(f) = \mu\big(\sum_{i \in \Lambda_l} \hat{c}_i \, Var_{B_i}(f)\big)$ and let $\mathcal{D}(f)$ be the Dirichlet form of the KA model. Then*

$$\mathcal{D}^l(f) \leq O(N^2) C(l,q) \mathcal{D}(f).$$

COROLLARY 5.2.46. *Fix $2 \leq k \leq d$ together with $q \in (0,1)$. Assume that it is possible to choose the mesoscopic scale l depending only on k,d,q in such a way that $\pi_l(d,k) \geq \pi_*$, where π_* is the constant appearing in Proposition 5.2.11. Then*

$$Var(f) \leq O(N^2) C(l,q) \mathcal{D}(f).$$

Equivalently

$$T_{rel}(q,L) \leq O(L^2) C(l,q).$$

PROOF OF THE COROLLARY. The first part of the corollary follows at once from Propositions 5.2.11 and 5.2.45. The second part is an immediate consequence of the first one and of the variational characterization of the relaxation time (see the beginning of Section 5.2). □

PROOF OF PROPOSITION 5.2.46. Recall definition (5.2.4) of the long range constraints $\hat{c}_i$ and let us consider one term $\mu\big(\hat{c}_i \mathrm{Var}_{B_i}(f)\big)$ appearing in the definition of $\mathcal{D}^l(f)$. Observe that $\hat{c}_i$ is measurable w.r.t. the σ-algebra $\mathcal{F}_i$. Conditionally on $\mathcal{F}_i$ and assuming that $\hat{c}_i = 1$, let $\mathcal{G}$ be a family of good paths for the vertex $i + \vec{e}_1 + \vec{e}_2 \in \mathbb{Z}_l^d$. Clearly $\hat{c}_i = 1$ implies that $G_{i,\gamma}$ occurs for each path $\gamma \in \mathcal{G}$. Hence, by applying Lemma 5.2.44 to each path in $\mathcal{G}$ we get

$$\mu\big(\mathrm{Var}_{B_i}(f)\,|\,\mathcal{F}_i\big) \le O(N)C(l,q)\,\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathcal{D}_{i,\gamma}$$

$$= O(N)C(l,q)\Big[\sum_{\substack{x,y\in\Lambda\\ \|x-y\|_1=1}} \mu\big(c_{xy}\big(\nabla_{xy}f\big)^2\,|\,\mathcal{F}_i\big)\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{(x,y)\subset V_{i,\gamma}\}}$$

$$+\sum_{x\in\partial\Lambda}\mu\big(\mathrm{Var}_x(f)\,|\,\mathcal{F}_i\big)\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{x\in V_{i,\gamma}\}}\Big]. \tag{5.2.10}$$

For a given bond $(x,y)\subset\Lambda$ (respectively $x\in\partial\Lambda$) let $j=j(x)$ be such that $B_j\ni x$ and let Π_j denote the $(\vec{e}_1,\vec{e}_2)$-plane in $\mathbb{Z}^d_l$ containing j. Since all the paths forming the family $\mathcal{G}$ belong to the plane Π_i, and are focused we immediately get that

$$\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{(x,y)\subset V_{i,\gamma}\}} = \mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\,\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{(x,y)\subset V_{i,\gamma}\}},$$

$$\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{x\in\partial V_{i,\gamma}\}} = \mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\,\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{x\in\partial V_{i,\gamma}\}}$$

where $\mathcal{R}_i$ is the set of points at distance at most $\sqrt{N}$ from the set $\{k: k=i+s(\vec{e}_1+\vec{e}_2),\, s\in\mathbb{N}\}$. Next, for $(x,y)\subset\Lambda$ (respectively $x\in\partial\Lambda$) such that $\|i-j\|_1\le\sqrt{N}$ we bound $\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{(x,y)\subset V_{i,\gamma}\}}$ (respectively $\frac{1}{|\mathcal{G}|}\sum_{\gamma\in\mathcal{G}}\mathbb{1}_{\{x\in\partial V_{i,\gamma}\}}$) by one. If instead $\|i-j\|_1>\sqrt{N}$ then we use the fact that the paths of $\mathcal{G}$ are almost edge-disjoint to bound from above both sums by $2/|\mathcal{G}|\le 2/\sqrt{N}$.

In conclusion, the first and second term inside the square bracket in the r.h.s. of (5.2.10) are bounded from above by

$$\sum_{\substack{x,y\in\Lambda\\ \|x-y\|_1=1}} \mu\big(c_{xy}\big(\nabla_{xy}f\big)^2\,|\,\mathcal{F}_i\big)\mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\big[\mathbb{1}_{\{\|i-j(x)\|_1\le\sqrt{N}\}}+\frac{2}{\sqrt{N}}\mathbb{1}_{\{\|i-j(x)\|_1>\sqrt{N}\}}\big]$$

and

$$\sum_{x\in\partial\Lambda}\mu\big(\mathrm{Var}_x(f)\,|\,\mathcal{F}_i\big)\mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\big[\mathbb{1}_{\{\|i-j(x)\|_1\le\sqrt{N}\}}+\frac{2}{\sqrt{N}}\mathbb{1}_{\{\|i-j(x)\|_1>\sqrt{N}\}}\big]$$

respectively. Clearly the same bounds hold for their average w.r.t. μ.

In order to conclude the proof it is enough to sum over i the above expressions and use the fact that, uniformly in $x\in\Lambda$,

$$\sum_{i\in\Lambda_l}\mathbb{1}_{\{j\in\Pi_i\}}\mathbb{1}_{\{j\in\mathcal{R}_i\}}\big[\mathbb{1}_{\{\|i-j(x)\|_1\le\sqrt{N}\}}+\frac{2}{\sqrt{N}}\mathbb{1}_{\{\|i-j(x)\|_1>\sqrt{N}\}}\big]\le O(N).$$

□

5.2.6. Completing the proof of Theorem 5.1.1. Using Corollary 5.2.46, the proof of Theorem 5.1.1 is complete if we can prove that for all $\pi^*<1$, for any given $q\in(0,1)$ and $2\le k\le d$ it is possible to choose $l=l(q,k,d)$ in such a way that

(i) the probability that any given $i\in\mathbb{Z}^d_l$ is (d,k)-good satisfies $\pi_l(d,k)\ge\pi_*$;

(ii) $C(l,q) \leq C(q)$ as $q \to 0$, where $C(q)$ is as in (5.1.3) and $C(l,q)$ satisfies (5.2.9).

Let us start by stating a key result on the probability of the set of frameable configurations.

PROPOSITION 5.2.47 (Probability of frameable configurations [**49**]). *Fix q and let $\mathcal{F}_q(l,d,j)$ be the probability that the cube $C_l = [l]^d$ is (d,j)-frameable. Then there exists $C > 0$ s.t. for $q \to 0$*

$$\mathcal{F}_q(l,d,j) \geq 1 - Ce^{-l_q/\Xi_{d,j}} \qquad \forall l_q \quad s.t. \quad \Xi_{d,k}(q) = O(l_q)$$

with

$$\Xi_{d,1}(q) := \left(\frac{1}{q}\right)^{1/d}$$

and

$$\Xi_{d,j}(q) := \exp_{(j-1)}\left(\frac{1}{q^{\frac{1}{d-j+1}}}\right) \qquad \forall j \in [2,d].$$

PROOF. The case $j = 1$ follows immediately from the definition of frameable configurations (see Definition 5.2.1). The cases $j \in [2,d]$ are proven in Section 2 of [**49**], see formula (34) [2] and (36), where the results are stated in terms of the parameter $s = j - 1$. Actually, the definition of frameable in [**49**] is more restrictive than our Definition 5.2.1. Indeed in [**49**] the frame that should be emptiable is composed by all the faces of dimension $j-1$ containing one corner of $\mathcal{C}_l$ (and not only those that contain the vertex $(1,\dots,1)$). However, since we only need a lower bound on the probability of being frameable we can directly use the results of [**49**]. □

Then, by using Proposition 5.2.47 and the Definition 5.2.2, we get that there exists $c > 0$ s.t. by choosing

$$l(q,k,d) = \exp_{(k-2)}\left(\frac{c}{q^{\frac{1}{d-j+1}}}\right) \quad \forall k \in [3,d]$$

and

$$l(q,2,d) = \frac{\log q}{q^{\frac{1}{d-1}}}$$

we get

$$\pi_l(d,k) \geq \left(1 - C\exp^{-l/\Xi_{d-1,k-1}}\right)^{ld}$$

which goes to 1 as $q \to 0$, and thus implies that condition (i) is satisfied for all $q \in (0,1)$ (since $\pi_l(d,k)$ is non decreasing with q) . Finally, it is immediate to verify that the above choice of l satisfies also condition (ii) for all $k \in [2,d]$.

□

[2]There is a misprint in formula (34) of [**49**]: in the exponential a minus sign is missing

CHAPTER 6

Questions

6.1. KCMs and bootstrap percolation in random environments

In this work we have seen an analysis of certain time scales in several models for the bootstrap percolation and the KCM, but many more questions are left open.

- Can we extend these results to general random mixtures of constraints, each given by a general update family (see [**39, 40**])?
- How do these time scales behave on other random graphs, such as the random regular graph, configuration model, or the Poisson Voronoi tessellation?
- What can we say about models with random external field, i.e., rather than having a constant q the equilibrium at each site x is determined by a random q_x?
- What can we say beyond the emptying time of a site? The question closer to the physical behavior of the system would, in fact, be the time for correlations to decay. Theorem 3.6.1 treats this problem, but the methods used in order to prove it do not apply to the other models studied here.
- Another type of question is when starting out of equilibrium, for example, with an independent product measure of parameter greater than q. How will hitting times behave in this case? And how long will the process take to mix?
- Can we say something about kinetically constraint lattice gas models (namely Kawasaki type) in random environments?

6.2. The Kob-Andersen model

- Can we find a lower bound on the relaxation time, showing that the constant $C(q)$ of Theorem 5.1.1 scales as in equation (5.1.3)?
- Improve the bound of [**20**] on the loss of correlation in this model. For the two dimensional case it is bounded there between $1/t$ and $(\log t)^5/t$, and we expect that the correct behavior is $1/t$. In higher dimension the lower bound is $1/t^{d/2}$, which is even further from the upper bound, and trying to match them will be an interesting problem.
- Hydrodynamic limit – some results are known for KCLG models, but only in the non-cooperative case [**29**]. Is it possible to establish a hydrodynamic limit for the Kob-Andersen model, and to which degenerate diffusion equation does it correspond?
- Understanding the dynamics of a single tagged particle. The diffusion constant is known to be positive, but the known bound decays extremely fast with q [**13**]. I

expect that the methods developed in chapter 5 could be used in order to find its correct behavior.

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APPENDIX A

Dynamical systems near a bifurcation point

Section 4.2 concerns with the analysis of a phase transition originating in the appearance of a new fixed point for a certain recurrence relation, i.e., a bifurcation. In this appendix, we will try to understand in a more general context the time scaling in systems of that type. Let us then consider a sequence of reals $\{x_n\}_{n=0}^{\infty}$, defined by the value x_0 and the a recursion formula for $n > 0$:

$$x_n = f(x_{n-1}). \tag{A1}$$

We will also fix now some positive $\delta < 1$, that will be used throughout this appendix as the window around the new fixed point in which we are interested.

First, we will study the time scaling at the bifurcation point, when the new fixed point is first created. In this case, we may expect f to be tangent to the identity function at the fixed point, so we will start our discussion with the following assumptions:

Assumption A1. *f has a fixed point y_0, such that for $y \in (y_0, y_0+\delta)$*

$$y - \underline{c}(y-y_0)^{\alpha} \le f(y) \le y - \bar{c}(y-y_0)^{\alpha},$$

for some $\alpha > 1$, $0 < \bar{c} \le \underline{c} < \delta^{-(\alpha-1)}$.

Assumption A2. $x_0 \in (y_0, y_0+\delta)$.

We first mention the following claim:

Claim A1. The sequence is decreasing and bounded from below by y_0.

Proof. By Assumption A1, $x_{n+1} < x_n$ whenever $x_n \in (y_0, y_0+\delta)$. Moreover:

$$\begin{aligned} x_{n+1} - y_0 &\ge x_n - y_0 - \underline{c}(x_n - y_0)^{\alpha} \\ &= (x_n - y_0)\left(1 - \underline{c}(x_n-y_0)^{\alpha-1}\right) \\ &\ge (x_n - y_0)\left(1 - \underline{c}\delta^{\alpha-1}\right) > 0. \end{aligned}$$

Therefore, since $x_0 \in (y_0, y_0+\delta)$ by assumption A2, the entire sequence is in the interval $(y_0, y_0+\delta)$, and it is decreasing. □

The following theorem will describe the asymptotic of the sequence:

Theorem A2. *Let $\{x_n\}$ be the sequence defined in equation* (A1)*, satisfying assumptions A1 and A2. Then*

$$y_0 + \underline{a}\,(n+n_0)^{-\frac{1}{\alpha-1}} \le x_n \le y_0 + \overline{a} n^{-\frac{1}{\alpha-1}},$$

where $\underline{a} = \left[(\alpha-1)(1-\delta)^{-\alpha}\underline{c}\right]^{-\frac{1}{\alpha-1}}$, $\overline{a} = \left[(\alpha-1)\overline{c}\right]^{-\frac{1}{\alpha-1}}$, *and* $n_0 = \frac{(x_0-y_0)^{1-\alpha}}{(\alpha-1)(1-\delta)^{-\alpha}\underline{c}}$ *are all positive constants.*

Proof. Let us first define a sequence $t_n = (x_n - y_0)^{1-\alpha}$, and note that t_n is positive for all n. Then using Claim A1 and Assumption A1, fixing $\underline{c}' = (\alpha-1)(1-\delta)^{-\alpha}\underline{c}$ and $\overline{c}' = (\nu-1)\overline{c}$:

$$\begin{aligned} t_n &= (f(x_{n-1}) - y_0)^{1-\alpha} \\ &\le (x_{n-1} - \underline{c}(x_{n-1}-y_0)^{\alpha} - y_0)^{1-\alpha} \\ &= \left(t_{n-1}^{\frac{1}{1-\alpha}} - \underline{c} t_{n-1}^{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} \\ &= t_{n-1}\left(1 - \underline{c} t_{n-1}^{-1}\right)^{1-\alpha} \\ &\le t_{n-1}\left(1 + \underline{c}' t_{n-1}^{-1}\right) \\ &= t_{n-1} + \underline{c}'; \end{aligned} \qquad \begin{aligned} t_n &= (f(x_{n-1}) - y_0)^{1-\alpha} \\ &\ge (x_{n-1} - \overline{c}(x_{n-1}-y_0)^{\alpha} - y_0)^{1-\alpha} \\ &= \left(t_{n-1}^{\frac{1}{1-\alpha}} - \overline{c} t_{n-1}^{\frac{\alpha}{1-\alpha}}\right) \\ &= t_{n-1}\left(1 - \overline{c} t_{n-1}^{-1}\right)^{1-\alpha} \\ &\ge t_{n-1}\left(1 + \overline{c}' t_{n-1}^{-1}\right) \\ &= t_{n-1} + \overline{c}'. \end{aligned}$$

We have used here the fact that, for any $0 < z < \delta < 1$, we can approximate $(1-z)^{1-\alpha}$ using its derivative at 0 and at δ:

$$-(1-\alpha) \le \frac{(1-z)^{1-\alpha} - 1}{z} \le -(1-\alpha)(1-\delta)^{-\alpha}.$$

We then also use $\underline{c} t_{n-1}^{-1} = (x_n - y_0)^{\alpha-1} < \delta^{\alpha-1} \le \delta$.

Finally,

$$\begin{aligned} x_n &= y_0 + t_n^{-\frac{1}{\alpha-1}} \\ &\le y_0 + \left((x_0-y_0)^{1-\alpha} + \overline{c}' n\right)^{-\frac{1}{\alpha-1}} \\ &\le y_0 + \overline{a} n^{-\frac{1}{\alpha-1}}; \end{aligned} \qquad \begin{aligned} x_n &\ge y_0 + \left((x_0-y_0)^{1-\alpha} + \underline{c}' n\right)^{-\frac{1}{\alpha-1}} \\ &= y_0 + \left(\underline{c}'\left(n + \tfrac{(x_0-y_0)^{1-\alpha}}{\underline{c}'}\right)\right)^{-\frac{1}{\alpha-1}} \\ &= y_0 + \underline{a}\,(n+n_0)^{-\frac{1}{\alpha-1}}. \end{aligned}$$

□

Next, we will be interested in the behavior near the bifurcation point, just before the new fixed point appears. For this purpose we will consider a family $\{x_n^{\varepsilon}\}_{n=0}^{\infty}$ of sequences, each defined by the value x_0^{ε} and the recursion formula for $n > 0$:

$$x_n^{\varepsilon} = f_{\varepsilon}(x_{n-1}), \tag{A2}$$

and assume:

Assumption A3. *There is a point* y_0 *such that for* $|y - y_0| < \delta$ *and* $\varepsilon < \varepsilon_0$

$$y - \underline{c}(y-y_0)^{2\alpha} - \varepsilon \le f_{\varepsilon}(y) \le y - \overline{c}(y-y_0)^{2\alpha} - \varepsilon,$$

for an integer $\alpha > 1$ *and positive constants* $\underline{c}, \overline{c}$.

Assumption A4. $0 < x_0 - y_0 < \delta$.

In order to study the asymptotic behavior of x_n^ε for small values of ε, we will need the following definition:

DEFINITION A3. The exit time $N_\delta(\varepsilon)$ is the minimal n such that $x_n^\varepsilon < y_0 - \delta$.

Replacing Claim A1 will be the following:

CLAIM A4. For all $\varepsilon < \varepsilon_0$ the exit time $N_\delta(\varepsilon)$ is finite, and for $n < N_\delta(\varepsilon)$ the sequence x_n^ε is decreasing.

PROOF. By assumption A3, for $n < N_\delta(\varepsilon)$, if $x_n^\varepsilon < y_0 + \delta$ then $x_{n+1}^\varepsilon < x_n^\varepsilon < y_0 + \delta$. Hence, the sequence remains in the interval $(y_0 - \delta, y_0 + \delta)$ as long as $n < N_\delta(\varepsilon)$. Since in this interval the sequence is decreasing, the results follows by Assumption A4. □

For our analysis, we will compare this sequence to the solution of the following differential equations, that will approximate $x_n^\varepsilon - y_0$:

$$\begin{gathered}\frac{\mathrm{d}\underline{\zeta}}{\mathrm{d}s} = -\underline{c}\,\underline{\zeta}^{2\alpha} - \varepsilon, \qquad \frac{\mathrm{d}\overline{\zeta}}{\mathrm{d}s} = -\overline{c}\,\overline{\zeta}^{2\alpha} - \varepsilon,\\ \underline{\zeta}\,(0) = z_0^\varepsilon = x_0^\varepsilon - y_0; \quad \overline{\zeta}\,(0) = z_0^\varepsilon = x_0^\varepsilon - y_0.\end{gathered}$$

The solution $\overline{\zeta}$ is strictly decreasing, and in particular one can define its inverse $\overline{t}: [-\infty, z_0^\varepsilon] \to [0, \infty]$, and $\overline{\tau}_n = \overline{t}\,(x_n^\varepsilon - y_0)$. $\underline{t}$ and $\underline{\tau}_n$ will be defined in the same manner. Note that they depend on ε, even though this dependence is omitted from the notation. The next lemma shows that the continuous crossing times $\overline{\tau}_n$ and $\underline{\tau}_n$ are close to the discrete one, namely n.

LEMMA A5. *For all* $n \le N_\delta(\varepsilon)$

$$(1 - \kappa_{\overline{c},\delta,\varepsilon})\, n \le \overline{\tau}_n \le \underline{\tau}_n \le (1 + \kappa_{\underline{c},\delta,\varepsilon})\, n,$$

where for all $c > 0$, $\kappa_{c,\delta,\varepsilon} = \max\left(C_4 \varepsilon^{2\alpha-1}, 2\alpha c \delta^{2\alpha-1}\right)$. C_4 *is a positive constant depending on* δ, c *and* ε_0 *given explicitly in the proof, and bounded when* δ *and* ε_0 *are not too big. For example, if* $\varepsilon_0 < 1$ *and* $c\delta^{2\alpha-1} < \frac{1}{2}$, $C_4 < (3 + 4^\alpha c)^{4\alpha}$.

PROOF. Let $z_n = x_n^\varepsilon - y_0$.

$$\begin{aligned}\underline{\tau}_n &= \underline{t}\left(f_\varepsilon\left(x_{n-1}^\varepsilon\right) - y_0\right) \le \underline{t}\left(z_{n-1} - \underline{c} z_{n-1}^{2\alpha} - \varepsilon\right) = \int_{z_0}^{z_{n-1} - \underline{c} z_{n-1}^{2\alpha} - \varepsilon} \frac{\mathrm{d}z}{-\underline{c} z^{2\alpha} - \varepsilon}\\ &= \underline{t}\,(z_{n-1}) - \int_{z_{n-1}}^{z_{n-1} - \underline{c} z_{n-1}^{2\alpha} - \varepsilon} \frac{\mathrm{d}z}{\underline{c} z_{n-1}^{2\alpha} + \varepsilon} - \int_{z_{n-1}}^{z_{n-1} - \underline{c} z_{n-1}^{2\alpha} - \varepsilon} \left(\frac{\mathrm{d}z}{\underline{c} z^{2\alpha} + \varepsilon} - \frac{\mathrm{d}z}{\underline{c} z_{n-1}^{2\alpha} + \varepsilon}\right)\\ &= \underline{\tau}_{n-1} + 1 - \int_{z_{n-1}}^{z_{n-1} - \underline{c} z_{n-1}^{2\alpha} - \varepsilon} \left(\frac{\mathrm{d}z}{\underline{c} z^{2\alpha} + \varepsilon} - \frac{\mathrm{d}z}{\underline{c} z_{n-1}^{2\alpha} + \varepsilon}\right).\end{aligned}$$

In order to study the error term, we use the following estimation:

CLAIM A6. Fix $w_0 \in (-\delta, \delta)$, and $c > 0$. Let

$$I = \int_{w_0 - cw_0^{2\alpha} - \varepsilon}^{w_0} \left(\frac{1}{cw^{2\alpha} + \varepsilon} - \frac{1}{cw_0^{2\alpha} + \varepsilon} \right) \mathrm{d}w.$$

Then

$$|I| \leq \kappa_{c,\delta,\varepsilon_0}.$$

PROOF. We will first consider the case in which the integral passes through 0, that is $0 < w_0 < cw_0^{2\alpha} + \varepsilon$. In this case,

$$\begin{aligned} w_0 &\leq w_0 \left(1 - cw_0^{2\alpha-1}\right) \left(1 - \delta^{2\alpha-1}\right)^{-1} \leq C_1 \varepsilon, \\ cw_0^{2\alpha} + \varepsilon &\leq \left[1 + C_2 \varepsilon^{2\alpha-1}\right] \varepsilon, \end{aligned}$$

for $C_1 = \left(1 - c\delta^{2\alpha-1}\right)^{-1}$ and $C_2 = c\left(1 - c\delta^{2\alpha-1}\right)^{-2\alpha}$.

We may then bound the nominator of the integrand for all $w \in [w_0 - cw_0^{2\alpha} - \varepsilon, w_0]$ by

$$\left| cw_0^{2\alpha} + \varepsilon - cw^{2\alpha} - \varepsilon \right| \leq cw_0^{2\alpha} + cw^{2\alpha} \leq C_3 \varepsilon^{2\alpha},$$

where $C_3 = \left(1 + C_2 \varepsilon_0^{2\alpha-1}\right)^{2\alpha} + C_1^{2\alpha}$.

For the denominator,

$$\left(cw^{2\alpha} + \varepsilon\right) \left(cw_0^{2\alpha} + \varepsilon\right) \geq \varepsilon^2.$$

Putting both estimations together

$$\begin{aligned} |I| &\leq \int_{w_0 - cw_0^{2\alpha} - \varepsilon}^{w_0} \left| \frac{cw_0^{2\alpha} + \varepsilon - cw^{2\alpha} - \varepsilon}{\left(cw^{2\alpha} + \varepsilon\right)\left(cw_0^{2\alpha} + \varepsilon\right)} \right| \\ &\leq \left(cw_0^{2\alpha} + \varepsilon\right) C_3 \varepsilon^{2\alpha-2} \leq C_4 \varepsilon^{2\alpha-1} \end{aligned}$$

for $C_4 = \left[1 + C_2 \varepsilon_0^{2\alpha-1}\right] C_3$.

Next, we consider the case in which the integral is over a positive interval, i.e., $w_0 \geq cw_0^{2\alpha} + \varepsilon$. We can bound the integrand using convexity – for all $w \in (w_0 - cw_0^{2\alpha} - \varepsilon, w_0)$

$$\frac{\frac{1}{cw^{2\alpha}+\varepsilon} - \frac{1}{cw_0^{2\alpha}+\varepsilon}}{w - w_0} \geq -\frac{2\alpha c w^{2\alpha-1}}{\left(cw^{2\alpha} + \varepsilon\right)^2}.$$

This implies that

$$\begin{aligned} I &\leq \left(cw_0^{2\alpha} + \varepsilon\right) \frac{2\alpha c w^{2\alpha-1}}{\left(cw^{2\alpha} + \varepsilon\right)^2} \left(w_0 - w\right) \\ &\leq 2\alpha c w^{2\alpha-1} \leq 2\alpha c \delta^{2\alpha-1}. \end{aligned}$$

We are left with the case $w_0 \leq -cw_0^{2\alpha} - \varepsilon$, which could be analyzed using the exact same argument as the previous one to obtain the result. □

Using this claim we can continue the estimation:

$$\underline{\tau}_n \leq \underline{\tau}_{n-1} + 1 + \kappa_{\underline{c},\delta,\varepsilon_0},$$

which proves the upper bound. The lower bound could be found by the exact same calculation, replacing $\underline{\tau}$ with $\overline{\tau}$ and $\underline{c}$ with $\overline{c}$. The result follows since $\overline{c} \leq \underline{c}$, thus $\overline{\tau}_n \leq \underline{\tau}_n$ by monotonicity of the integral. □

We are now ready to formulate the final result:

THEOREM A7. *Fix the family of sequences (indexed by ε) defined in equation* (A2) *satisfying assumptions A3 and A4. Consider the exit times $N_\delta(\varepsilon)$ (see Definition A3). Let $\overline{I} = \int_{-\infty}^{\infty} \frac{du}{\overline{c}u^{2\alpha}+1}$, $\underline{I} = \int_{-\infty}^{\infty} \frac{du}{\underline{c}u^{2\alpha}+1}$, and $\kappa_{\delta,0} = \max\left(\kappa_{\underline{c},\delta,0}, \kappa_{\overline{c},\delta,0}\right)$ for κ given in Lemma A5. Assume further that $\kappa_{\delta,0} < 1$. Then:*

$$0 < \frac{\frac{1}{2}\underline{I}}{(1+\kappa_{\delta,0})} \leq \liminf_{\varepsilon\to 0} \frac{N_\delta(\varepsilon)}{\varepsilon^{-1+\frac{1}{2\alpha}}} \leq \limsup_{\varepsilon\to 0} \frac{N_\delta(\varepsilon)}{\varepsilon^{-1+\frac{1}{2\alpha}}} \leq \frac{\overline{I}}{(1-\kappa_{\delta,0})} < \infty.$$

The factor of $\frac{1}{2}$ in front of $\underline{I}$ could be removed when $\varepsilon^{-\frac{1}{2\alpha}}\left(x_0^\varepsilon - y_0\right) \to \infty$ as $\varepsilon \to 0$ (e.g., when $x_0^\varepsilon - y_0$ is positive uniformly in ε).

PROOF. This theorem is a direct consequence of the fact that ζ shows an $\varepsilon^{-1+\frac{1}{2\nu}}$ time scaling behavior. First, note that

$$\overline{\tau}_{N_\delta(\varepsilon)-1} \leq \overline{t}(-\delta) \leq \underline{t}(-\delta) \leq \underline{\tau}_{N_\delta(\varepsilon)}.$$

We will then be interested in finding $\overline{t}(-\delta)$, $\underline{t}(-\delta)$:

$$t(-\delta) = \int_{z_0^\varepsilon}^{-\delta} \frac{\mathrm{d}z}{-cz^{2\alpha}-\varepsilon} = \varepsilon^{-1+\frac{1}{2\alpha}} \int_{z_0^\varepsilon}^{-\delta} \frac{\varepsilon^{-\frac{1}{2\alpha}}\mathrm{d}z}{-c\left(z\varepsilon^{-\frac{1}{2\alpha}}\right)^{2\alpha}-1} = \varepsilon^{-1+\frac{1}{2\alpha}} \int_{-\varepsilon^{-\frac{1}{2\alpha}}z_0^\varepsilon}^{\varepsilon^{-\frac{1}{2\alpha}}\delta} \frac{\mathrm{d}u}{cu^{2\alpha}+1},$$

where for $\overline{t}$ one should take $c = \overline{c}$, and $c = \underline{c}$ for $\underline{c}$.

All that is left is to use Lemma A5, finding

$$\frac{\underline{t}(-\delta)}{(1+\kappa_{\delta,\varepsilon})} \leq N_\delta(\varepsilon) \leq 1 + \frac{\overline{t}(-\delta)}{(1-\kappa_{\delta,\varepsilon})},$$

which, since the integrals defining $\overline{I}$ and $\underline{I}$ converge, concludes the proof. □

REMARK A8. When f_ε satisfies not only Assumption A3, but also

$$f_\varepsilon(y) = y - c\left(y-y_0\right)^{2\alpha} - \varepsilon + o\left(\left(y-y_0\right)^{2\alpha}\right) + o\left(\varepsilon\right),$$

we can consider δ_ε that goes to 0 with ε, e.g., $\frac{1}{|\log\varepsilon|}$, so that $\kappa_{\delta,0}$ will converge to 0 as well. In this case, we may choose $\overline{c}_\delta$ and $\underline{c}_\delta$ that converge to c, and thus Theorem A7 will give the limit of $\frac{N_\delta(\varepsilon)}{\varepsilon^{-1+\frac{1}{2\alpha}}}$, rather than just bounds on its limsup and liminf. Such a direct application of the theorem, however, forces us to choose an initial condition x_0^ε that converges to y_0 as ε goes to 0. To overcome this issue, we can use the estimation above with a fixed δ until x_n reaches δ_ε, which happens at n of order $\int_{z_0}^{\delta_\varepsilon} \frac{\mathrm{d}z}{-cz^{2\alpha}-\varepsilon} \ll \varepsilon^{-1+\frac{1}{2\alpha}}$. Then restart the dynamics using the estimation with δ_ε until reaching $-\delta_\varepsilon$, which takes an order $\varepsilon^{-1+\frac{1}{2\alpha}}$ of steps, and then using again the estimation for our fixed δ show that the number of steps required to reach $-\delta$ is much smaller than $\varepsilon^{-1+\frac{1}{2\alpha}}$. This would yield

$$\lim_{\varepsilon\to 0} \frac{N_\delta(\varepsilon)}{\varepsilon^{-1+\frac{1}{2\nu}}} = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{cu^{2\nu}+1}.$$