Noncooperative models of kinetically constrained lattice gases

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ABSTRACT. We study a family of conservative interacting particle systems with degenerate rates called noncooperative kinetically constrained lattice gases. We prove for all models in this family the diffusive scaling of the relaxation time, the positivity of the diffusion coefficient, and the positivity of the self-diffusion coefficient.

1. Introduction

Kinetically constrained lattice gases are interacting particle systems introduced by physicists in order to better understand glassy materials (see, e.g., [19, 24]). The basic underlying hypothesis behind these models is that glassy behavior is a dynamic effect, and the role of interactions is irrelevant. Under this hypothesis, we can explain why glasses are rigid using the *cage effect*—even though their microscopic structure is amorphous, glasses at low temperatures have a very high density, and molecules are unable to move since they are blocked by neighboring molecules.

In order to model this effect, we consider the lattice $\mathbb{Z}^d$, describing a coarse graining of the glass. Each site, corresponding to some region in the glass, could be either *occupied* or *empty*, representing dense or dilute zones. We think of the glass as very dense, so the small parameter q will be the ratio of *empty* sites.

The dynamics of kinetically constrained lattice gases is conservative—particles could jump between neighbors, turning an occupied site empty and a neighboring empty site occupied. However, not all jumps are allowed—in order to imitate the cage effect, when the local neighborhood of a particle is too dense it is blocked. That is, particles are only able to move under a certain constraint, satisfied when there are many vacancies nearby. Different kinetically constraint lattice gases are given by different choices of this constraint, namely, different interpretations of the neighborhood being "too dense".

To fix an idea, consider a one dimensional model introduced in [2] (see Example 2.1), where a particle is allowed jump to an empty neighbor, if it has at least two empty neighbors before or after the jump (including the site it jumped to/from). Note that if a particle is allowed to jump, than it is also allowed to jump back immediately after. This is a property we require for all kinetically constrained models, and it guarantees a noninteracting equilibrium.

It is instructive to compare these models to another family of interacting particle systems, the nonconservative kinetically constrained models (see, e.g., [10]). In those models, rather than jumping between sites, particles appear and disappear under the constraint. These models are in general simpler to analyze, and, at least in one and two dimensions, we have

FIGURE 1.1. This figure shows how, in the model described in Example 2.1, a mobile cluster can propagate. The mobile cluster here consists of the two empty sites, and after a sequence of 1 allowed transitions it is moved one step to the right. See Example 3.14.

a relatively good understanding of their behavior [21, 20, 15, 14, 13, 12]. In fact, one can identify a handful of universality classes describing the properties of a kinetically constrained model. Moreover, a simple criterion allows us to determine, given any translation invariant local constraint, to which universality class the model belongs. In the case of conservative kinetically constrained lattice gases, however, only a few specific models have been analyzed [2, 6, 11, 23, 29, 22, 4, 9, 25], and no general results are available.

We distinguish between two types of kinetically constrained lattice gases—cooperative and noncooperative. In a cooperative dynamics, any large scale change in the configuration forces many particles to move in order to "free up" space. In noncooperative models, small empty clusters can move around the lattice, without requiring any cooperation from other sites near them. Consider the example introduced above. Figure 1.1 shows how, in two allowed transitions, two neighboring vacancies can propagate to the right, no matter what the occupation is elsewhere. We say that these vacancies form a *mobile cluste*r. Noncooperative models are those where a mobile cluster exists, and cooperative models are models where no finite set of vacancies can propagate without any outside help. See Definition 3.13.

One simple implication of the presence of a mobile cluster is that the critical density of the model is 1 (equivalently, the critical value of q is 0). This means that for any $q > 0$, in an infinite system, there exists with probability 1 a sequence of allowed transitions in the end of which the origin (or any other arbitrary vertex) is empty. Indeed, since a mobile cluster consists of some fixed number of vacancies, if $q > 0$ there will almost surely be an empty mobile cluster somewhere in $\mathbb{Z}^d$. We can then move this cluster until one of its vacancies reaches the origin. In cooperative models identifying the critical density is more complicated. The only cooperative kinetically constrained lattice gas studied in the mathematics literature is the Kob-Andersen model [29], where the critical density is also 1; but in general cooperative models may have critical densities which are strictly smaller.

Close to criticality, when $q \ll 1$, most sites are occupied, and the constraint is rarely satisfied. The dynamics then tends to slow down, making typical time scales diverge. We will try to understand how significant this effect is. In the unconstrained model (namely the simple exclusion process), time scales diffusively, as the square of the distance:

$$\text{typical time} \approx C \times \text{ typical distance}^2.$$

We will see that noncooperative models are also diffusive—the constraint may affect the coefficient C, but the exponent remains 2. This will be done in four different contexts, giving different interpretations to “typical time” and “typical distance”.

The first time scale we study is the *relaxation time*, describing the time scale over which correlations are lost. Consider some observable f depending on the configuration, and measure it at time 0 and at time t. In some cases, the correlation between these two quantities, f_0 and f_t, decreases exponentially fast with t—

$$\text{Corr}(f_0, f_t) \leq e^{-t/\tau}.$$

The best (i.e. smallest) coefficient τ for which this decay hold *uniformly* (i.e. for all f) is the relaxation time. In general, the relaxation time can be infinite, and this is in fact the case for kinetically contrained lattice gases on the infinite lattice. In sections 4 and 5 we study the relaxation time on a *finite* box, of length L. We will see that the relaxation time is proportional to L^2, and that the corresponding coefficient diverges as a power law for small values of q.

In Section 6 we study the diffusion coefficient associated with the dynamics. This coefficient, generally speaking, describes the large scale evolution of the density profile. Consider for example a one dimensional model defined on a large interval $\{1, \dots, L\}$. Assume that the initial configuration approximates some given density profile $\rho_0 : [0,1] \to [0,1]$. Roughly speaking, this means that the number of particles in an interval $\{x - l/2, \dots, x, \dots, x + l/2\}$ of "medium" length (i.e. $1 \ll l \ll L$) is close to $l\rho(x/L)$. Then, when the system is diffusive, we expect the configuration at a later time t to approximate the same profile ρ_0 if $t \ll L^2$ (before the diffusive time scale), some evolving profile $\rho(t/L^2, \cdot)$ when t is of the order L^2 (in the diffusive time scale), and a uniform profile when $t \gg L^2$ (after the diffusive time scale). Moreover, the evolution in the diffusive scale is given by a *diffusion equation*

$$\partial_\tau \rho(\tau, \xi) = \partial_\xi \, D(\rho(\tau, \xi)) \partial_\xi \rho(\tau, \xi).$$

The *diffusion coefficient* D tells us, within the diffusive scale, how fast the density profile changes. In particular, if $D = 0$ the density profile does not evolve in diffusive time scales. When this picture indeed describes the behavior of the model, we say that it converges to a hydrodynamic limit in the diffusive scale. This hydrodynamic limit is given by the diffusion equation above. For a more complete discussion see, e.g., [17].

Proving rigorously converges to a hydrodynamic limit is not an easy task, accomplished only for one example of a kinetically constrained lattice gas [11, 3]. In fact, it cannot hold in full generality—a configuration such as the one shown in Figure 1.2 approximates the profile

$$\rho_0(x) = \begin{cases} 1 & x \leq L/2, \\ 2/3 & x > L/2. \end{cases}$$

At the same time, the configuration is blocked, namely, no particle is allowed to jump. Thus, the density profile remains fixed, and cannot converge to a hydrodynamic limit. This initial configuration, though, is very specific, and one may still hope that, by restricting to a more

FIGURE 1.2. We see here a blocked configuration for the model in Example 2.1—in the left half all sites are filled, while in the right half one in every three sites is empty. No particle could jump to an empty site, hence the configuration is blocked. In particular, it cannot converge to the hydrodynamic limit.

generic initial state, the dynamics will convergence to a hydrodynamic limit. This is proven in [11] for the model they study, but a general proof seems to be very difficult.

Still, even without proving convergence, studying the diffusion coefficient is an interesting problem, allowing us to obtain a plausible candidate for the hydrodynamic limit [1, 28, 25]. Moreover, the strategy of [25] shows convergence to a hydrodynamic limit in a "soft" sense whenever the model is rotation invariant. In particular, a strictly positive diffusion coefficient is a good indication that the density profile evolves over diffusive time scales. In Section 6 we show that the diffusion coefficient of noncooperative kinetically constrained lattice gases is indeed positive, and that it decays at most polynomially fast for small q.

The last interpretation of "typical time" and "typical distance" we consider is perhaps the most intuitive. Assume that the initial configuration has a particle at the origin called the *tracer* (but otherwise sampled from equilibrium). One may think of the tracer as playing the role of the pollen grain in Brown's famous experiment. We then follow its motion, and ask what is the time it would typically take in order to cross a certain distance. Diffusive scaling means that this time scales as the square of the distance. A general argument of [16] shows a much stronger result—under diffusive scaling, the path of the tracer converges to a Brownian motion. The variance of this Brownian motion is called the *self diffusion* D_s, and when it is strictly positive the Brownian motion in nondegenerate, i.e., the relevant time scale is indeed diffusive.

All quantities mentioned above have variational characterizations, involving infima or suprema over local functions, see equations (4.2), (6.1), and (7.1). These formulations allow us to analyze them using canonical path arguments, which in the lack of attractivity have proven extremely useful in the study of kinetically constrained models and kinetically constrained lattice gases (see, e.g., [5, 6, 2, 22, 4, 25]). In this paper, following [22, 9, 25], we formulate these argument in the language of *multistep moves*, see Definition 3.1. These are sequences of transitions, each allowed for the dynamics, leading to some desired final configuration.

1.1. **Structure of the paper.** In Section 2 we set up some of the notation, and define kinetically constrained lattice gases. We also introduce two examples that will be referred to throughout the paper.

Is Section 3 we introduce the notion of a multistep move and its basic properties. We then use this notion in order to precisely define of a mobile cluster and noncooperative models. Finally, we provide a slightly weaker characterization of noncooperative models.

The two following sections discuss the relaxation time in two different settings—Section 4 concerns with systems connected to a reservoir, while in Section 5 we analyze closed systems. The result of Section 4 shows diffusivity of the relaxation time in all noncooperative models. It generalizes [2], and the proof uses the same strategy in a wider context and in the language of multistep moves.

Studying the relaxation time in closed systems is much more involved. This problem was analyzed for one noncooperative model in [11], proving diffusive scaling if the density is low enough or when adding a small perturbation violating the constraint. The same model was later considered in [23], where diffusivity was proven for all densities and with no perturbation. Here, in Section 5, we generalize the result of [23] to some class of noncooperative models. The proof of the result uses a completely different strategy—while [23] relies on specific combinatorial details of the model they study, the proof here only uses general properties of mobile clusters. This new strategy allows us to obtain a result in a wider context.

In Section 6 we show that the diffusion coefficient is positive for all noncooperative models. In order to achieve that, we introduce a new comparison argument using multistep moves (Lemma 6.4). We then construct an auxiliary dynamics which on one hand can be compared to the kinetically constrained gas in question, and on the other hand possesses a special property allowing us to calculate its diffusion coefficient explicitly.

The positivity of the self-diffusion coefficient for all noncooperative models (in dimension 2 and above) is proven in Section 7. The proof applies a strategy similar to [27, II.6], using a multistep move in order to compare the kinetically constrained lattice gas to a random walk.

We conclude with open problems that this work suggests.

2. The model

2.1. **Notation.** In order to simplify the exposition of the model, we start by defining some of the notation we use.

- For $n \in \mathbb{N}$, we denote $[n] = \{1, \dots, n\}$.
- We will consider models defined either on $\mathbb{Z}^d$, a finite box $[L]^d$ for $L \in \mathbb{N}$, or the torus $\mathbb{Z}^d/L\mathbb{Z}^d$. We denote by $\{e_\alpha\}_{\alpha=1}^d$ the standard basis, and we say that two sites x and y are neighbors, denoted $x \sim y$, if $x - y \in \{\pm e_1, \dots, \pm e_d\}$. The *boundary* of a set $\Lambda \subset \mathbb{Z}^d$, denoted $\partial\Lambda$, is the set of sites in Λ that have a neighbor outside Λ.
- For any finite sequence of sites $x_1, \dots, x_n$, we denote by $\sigma = (x_1, \dots, x_n)$ the corresponding cyclic permutation, i.e., for any site y

$$\sigma(y) = \begin{cases} x_{k+1} & \text{if } y = x_k \text{ for } k \in [n-1], \\ x_1 & \text{if } y = x_n, \\ y & \text{otherwise.} \end{cases}$$

For a fixed site x we denote by τ_x the permutation on $\mathbb{Z}^d$ given by a translation by x, i.e., for any site $y \in \mathbb{Z}^d$

$$\tau_x(y) = y + x.$$

- A configuration is an element η of $\Omega = \Omega_\Lambda = \{0,1\}^\Lambda$, where Λ is either $\mathbb{Z}^d$, $[L]^d$, or the torus. We say that a site $x \in \Lambda$ is *empty* if $\eta(x) = 0$ and *occupied* if $\eta(x) = 1$.
- For $\eta \in \Omega$ and a site x we define η^x to be the configuration η after flipping the occupation at x.
- For $\eta \in \Omega$ and two sites x and y we define $\eta^{x,y}$ to be the configuration η after exchanging the occupation values at x and y.
- For $\eta \in \Omega$ and a permutation σ, we define $\sigma\eta$ to be the configuration after applying σ, i.e., for any site y

  $$(\sigma\eta)(y) = \eta(\sigma^{-1}(y)).$$

  In particular, for any two sites x and y we can write $\eta^{x,y} = (x,y)\eta$.
- For a $f : \Omega \to \mathbb{R}$ and two sites x and y,

  $$\nabla_x f(\eta) = f(\eta^x) - f(\eta),$$
  $$\nabla_{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta).$$

- For a $f : \Omega \to \mathbb{R}$ and a permutation σ, we define the function σf as

  $$\sigma f(\eta) = f(\sigma^{-1}\eta).$$

Finally, we note that throughout the paper C represents a generic positive constant, that may depend only on the model (dimension and constraints), and in particular does *not* depend on the parameter q.

2.2. **Kinetically constrained lattice gases.** Kinetically constrained lattice gases are interacting particle systems, defined on $\mathbb{Z}^d$, with generator $\mathcal{L}$ acting on any local function $f : \Omega \to \mathbb{R}$ as

$$\mathcal{L}f(\eta) = \sum_{x \sim y} c_{x,y}(\eta) \nabla_{x,y} f(\eta). \tag{2.1}$$

The rates $c_{x,y}$ must have the following properties:

(1) For any $x \sim y$ and $\eta \in \Omega$, $c_{x,y}(\eta) \in \{0\} \cup [1, c_{\text{max}}]$ for some $c_{\text{max}} \geq 1$.
(2) The rate $c_{x,y}$ depends only on the configuration outside x and y.
(3) The rates are nondegenerate, i.e., for any edge $x \sim y$ there exists a configuration $\eta \in \Omega$ such that $c_{x,y}(\eta) \geq 1$ and a configuration $\eta' \in \Omega$ such that $c_{x,y}(\eta) = 0$.
(4) For fixed x and y, the rate is a decreasing function of η, i.e., emptying sites could only speed up the dynamics.
(5) The model is homogeneous: $c_{x,y}(\eta) = c_{\tau_z(x),\tau_z(y)}(\tau_z\eta)$ for any $\eta \in \Omega$ and $x, y, z \in \mathbb{Z}^d$.

(6) The rates have finite range, i.e., $c_{x,y}$ depends only on the occupation of the sites in some box $x + [-R, R]$, where R is called the *range*.

Sometimes we refer to the rate $c_{x,y}$ as the *constraint* (having in mind the case $c_{\max} = 1$), and say that the constraint is satisfied when $c_{x,y} \geq 1$ and not satisfied when $c_{x,y} = 0$.

We may also consider the model on a subset of the lattice $\Lambda \subset \mathbb{Z}^d$ (usually $[L]^d$) by thinking of the sites outside Λ as empty. The generator has the same form as (2.1), with sum taken over $x, y \in \Lambda$. The constraint $c_{x,y}(\eta)$ for $\eta \in \{0,1\}^\Lambda$ is then defined to be $c_{x,y}(\overline{\eta})$, where $\overline{\eta} \in \{0,1\}^{\mathbb{Z}^d}$ is the configuration which equals η on Λ and 0 outside Λ. Theses are the *empty boundary conditions*. The *occupied boundary conditions* are defined analogously. Finally, *periodic boundary conditions* are defined when considering the model on the torus. The constraint $c_{x,y}(\eta)$ for $\eta \in \{0,1\}^{\mathbb{Z}^d/L\mathbb{Z}^d}$ is then given by $c_{x,y}(\overline{\eta})$ with $\overline{\eta}(x) = \eta(x \bmod L^d)$.

Under the assumptions above, the dynamics is reversible with respect to a product measure for any density in $[0,1]$. We refer to this measure as the *equilibrium measure* (at a given density). The density of *empty* sites is denoted by $q \in [0,1]$, so the equilibrium measure $\mu = \mu_q$ assigns to each site an independent Bernoulli random variable with parameter $1 - q$.

On a finite box $\Lambda = [L]^d$, we may consider a kinetically constrained lattice gas with reservoir on the boundary. This model is defined by the generator $\mathcal{L}_{\mathrm{r}}$ operating on any local function $f : \Omega \to \mathbb{R}$ as

$$\mathcal{L}_{\mathrm{r}} f(\eta) = \sum_{\substack{x,y \in \Lambda \\ x \sim y}} c_{x,y}(\eta) \nabla_{x,y} f(\eta) + \sum_{x \in \partial\Lambda} c_x \nabla_x f(\eta), \tag{2.2}$$

where $c_x(\eta) = q\eta(x) + (1-q)(1-\eta(x))$. Note that $c_x(\eta)$ is chosen such that the process remains reversible with respect to μ.

2.3. **Examples.** Throughout the paper, we will refer to two fundamental examples:

Example 2.1. The 1 dimensional model, with constraint

$$c_{x,x+1}(\eta) = \begin{cases} 1 & \text{if } \eta(x-1) = 0 \text{ or } \eta(x+2) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This model was introduced in [2], and further studied in [23]. In [11] a slight variation was introduced, where the rate $c_{x,x+1}$ equals 2 if both $\eta(x-1)$ and $\eta(x+2)$ are empty. This difference is of no importance to the analysis in this paper, but it does introduce a significant simplification in proving the convergence to a hydrodynamic limit.

Example 2.2. The 2 dimensional model with constraint

$$c_{x,x+e_\alpha}(\eta) = \begin{cases} 1 & \text{if } \eta(x-e_\alpha) = 0 \text{ or } \eta(x+2e_\alpha) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $\alpha \in \{1,2\}$.

This could be seen is a generalization of Example 2.1, also studied in [2].

3. Multistep moves

The main tool we use in this paper are *multistep moves*, which are sequences of transitions allowed for the dynamics, taking us from one configuration to the other. This formulation, used in [22, 9, 25], makes the application of canonical path methods more transparent.

A multistep move provides, for η in some fixed set of configuration (the domain), a sequence of transitions that are allowed for the dynamics. That is, at each step t it will tell us which edge to exchange in order to move from the configuration η_t to η_{t+1}. In order for the move to be valid, in all exchanges the constraint must be satisfied. This is expressed in the following definition:

Definition 3.1 (Multistep move)**.** For fixed $T > 0$, a T-step move M defined on $\operatorname{Dom} M \subseteq \Omega$ is a triple $\left((\eta_t)_{t=0}^{T}, (x_t)_{t=0}^{T-1}, (e_t)_{t=0}^{T-1}\right)$; where $(\eta_t)_{t=0}^{T}$ is a sequence of functions $\eta_t : \operatorname{Dom} M \to \Omega$, $(x_t)_{t=0}^{T-1}$ is a sequence of functions $x_t : \operatorname{Dom} M \to \mathbb{Z}^d$, and $(e_t)_{t=0}^{T-1}$ is a sequence of functions $e_t : \operatorname{Dom} M \to \{\pm e_1, \ldots, \pm e_d\}$. The move must satisfy the following properties:

(1) For any $\eta \in \operatorname{Dom} M$, $\eta_0(\eta) = \eta$.
(2) For any $\eta \in \operatorname{Dom} M$ and $t \in \{0, \ldots, T-1\}$,
   (a) on the infinite lattice or a finite box with no reservoirs,

$$\eta_{t+1}(\eta) = \eta_t(\eta)^{x_t(\eta), x_t(\eta)+e_t(\eta)} \text{ and } c_{x_t(\eta), x_t(\eta)+e_t(\eta)}(\eta_t(\eta)) = 1.$$

   (b) on a finite box Λ with reservoirs, either

$$\eta_{t+1}(\eta) = \eta_t(\eta)^{x_t(\eta), x_t(\eta)+e_t(\eta)} \text{ and } c_{x_t(\eta), x_t(\eta)+e_t(\eta)}(\eta_t(\eta)) = 1,$$

   or

$$\eta_{t+1}(\eta) = \eta_t(\eta)^{x_t(\eta)} \text{ and } x_t(\eta) \in \partial\Lambda.$$

When context allows we omit, with some abuse of notation, the explicit dependence on η (writing η_t, x_t, e_t rather than $\eta_t(\eta), x_t(\eta), e_t(\eta)$).

We continue with several basic notions related to multistep moves.

Definition 3.2 (Information loss)**.** Consider a T-step move $M = ((\eta_t), (x_t), (e_t))$ and $t \in \{0, \ldots, T\}$. The *loss of information at time* t is defined as

$$2^{\operatorname{Loss}_t M} = \sup_{\eta', x', e'} \#\left\{\eta \in \operatorname{Dom} M \text{ such that } \eta_t(\eta) = \eta', x_t(\eta) = x' \text{ and } e_t(\eta) = e'\right\},$$

where the supremum is taken over $\eta' \in \operatorname{Dom} M$, $x' \in \mathbb{Z}^d$ and $e' \in \{\pm e_1, \ldots, \pm e_d\}$. We also define

$$\operatorname{Loss} M = \sup_t \operatorname{Loss}_t M.$$

That is, for given t, η', x', e' there are at most $2^{\operatorname{Loss} M}$ possible configurations $\eta \in \operatorname{Dom} M$ for which $\eta_t = \eta'$, $x_t = x'$ and $e_t = e'$.

Definition 3.3 (Energy barrier)**.** Consider a T-step move $M = ((\eta_t), (x_t), (e_t))$ for a kinetically constrained lattice gas defined on a finite box Λ with reservoirs on the boundaries. The energy barrier is

$$\mathrm{EB}(M) = \sup_{t\in\{0,\dots T\}} \sup_{\eta\in \mathrm{Dom}\,\Omega} \left(\#\left\{\text{empty sites in } \eta_t\right\} - \#\left\{\text{empty sites in } \eta\right\}\right).$$

Note that, since $\eta_0 = \eta$, $\mathrm{EB}(M) \geq 0$.

Definition 3.4 (Composition of multistep moves)**.** Fix a T_1-step move $M_1 = ((\eta_t^1), (x_t^1), (e_t^1))$ and a T_2-step move $M_2 = ((\eta_t^2), (x_t^2), (e_t^2))$ such that for any $\eta \in \mathrm{Dom}\, M_1$, $\eta_{T_1}^1(\eta) \in \mathrm{Dom}\, M_2$. Then their composition $M_2 \circ M_1$ is the T-step move $M = ((\eta_t), (x_t), (e_t))$, with $T = T_1 + T_2$ and $\mathrm{Dom}\, M = \mathrm{Dom}\, M_1$ given by

$$\eta_t(\eta) = \begin{cases} \eta_t^1(\eta) & \text{if } t \in \{0,\dots,T_1\}, \\ \eta_{t-T_1}^2(\eta_T^1(\eta)) & \text{otherwise,} \end{cases}$$

$$x_t(\eta) = \begin{cases} x_t^1(\eta) & \text{if } t \in \{0,\dots,T_1\}, \\ x_{t-T_1}^2(\eta_T^1(\eta)) & \text{otherwise,} \end{cases}$$

$$e_t(\eta) = \begin{cases} e_t^1(\eta) & \text{if } t \in \{0,\dots,T_1\}, \\ e_{t-T_1}^2(\eta_T^1(\eta)) & \text{otherwise.} \end{cases}$$

Definition 3.5 (Associated permutation)**.** We consider here a model with no reservoirs. Fix a T-step move $M = ((\eta_t), (x_t), (e_t))$ and $\eta \in \mathrm{Dom}\, M$. Then the *associated permutation* σ is a permutation on the sites of $\mathbb{Z}^d$ given by the product of transpositions $(x_{T-1}, x_{T-1} + e_{T-1})(x_{T-2}, x_{T-2} + e_{T-2}) \dots (x_0, x_0 + e_0)$.

We say that the move M is *compatible with a permutation* σ if, for any $\eta \in \mathrm{Dom}\, M$, the associated permutation is σ.

Observation 3.6. Fix a T-step move $M = ((\eta_t), (x_t), (e_t))$ and $\eta \in \mathrm{Dom}\, M$. Then $\eta_T = \sigma\eta$, i.e., for any $x \in \mathbb{Z}^d$,

$$\eta_T(\sigma(x)) = \eta(x).$$

Observation 3.7. Consider two multistep moves M_1 and M_2. Assume that M_1 is compatible with a permutation σ_1 and M_2 with a permutation σ_2. If $M_2 \circ M_1$ is well defined, then it is compatible with $\sigma_2\sigma_1$.

Definition 3.8 (Deterministic move)**.** A T-step move $M = ((\eta_t), (x_t), (e_t))$ is called *deterministic* if the sequences $(x_t)_{t=0}^{T-1}$ and $(e_t)_{t=0}^{T-1}$ do not depend on η, that is, for any $\eta, \eta' \in \mathrm{Dom}\, M$ and any $t \in \{0,\dots,T-1\}$, $x_t(\eta) = x_t(\eta')$ and $e_t(\eta) = e_t(\eta')$. Note that a deterministic move is always compatible with a permutation, and has 0 loss of information.

Observation 3.9. Consider a deterministic T-step move $M = ((\eta_t), (x_t), (e_t))$ compatible with a permutation σ. The there exists an inverse move M^{-1} with domain

$$\operatorname{Dom} M^{-1} = \{\eta \in \Omega : \sigma\eta \in \operatorname{Dom} M\},$$

which is a T-step move compatible with σ^{-1}.

These are the general definitions and basic properties of multistep moves. We now continue with a few definitions related to the noncooperative nature of the model. In each definition, we will describe a move that changes the configuration in a desired way without "disturbing" too many sites, under the condition that there is a mobile cluster near by. The way in which we change the configuration is given by the permutation the move is compatible with. The fact that we do not want to disturb many sites is expressed in the fact that all x_t's are restricted to some given box. The requirement that a mobile cluster is available is expressed in the domain of the multistep move.

The first move we define will allow us to move a mobile cluster $\mathcal{C}$ on the lattice:

Definition 3.10 (Translation move)**.** Fix a finite set $\mathcal{C} \subset \mathbb{Z}^d$, $l > 0$, $e \in \{\pm e_1, \ldots, \pm e_d\}$ and $x \in \mathbb{Z}^d$. A *translation move* in $[-l, l]^d$ of the cluster $x + \mathcal{C}$ in the direction e is a T_{Tr}-step move $\text{Tr}_e(x + \mathcal{C})$ satisfying:

(1) $\operatorname{Dom} \text{Tr}_e(x + \mathcal{C}) = \{\eta \in \Omega : x + \mathcal{C} \text{ is empty}\}$
(2) $\text{Tr}_e(x + \mathcal{C})$ is a deterministic move, compatible with a permutation σ.
(3) $\sigma(x + y) = x + y + e$ for any $y \in \mathcal{C}$.
(4) For all $t \in \{0, \ldots, T - 1\}$, $x_t \in x + [-l, l]^d$ and $x_t + e_t \in x + [-l, l]^d$.

For brevity, we may write $\text{Tr}_{\pm\alpha}$ rather than $\text{Tr}_{\pm e_\alpha}$.

Observation 3.11. Fix a $\mathcal{C} \subset \mathbb{Z}^d$, $l > 0$, $e \in \{\pm e_1, \ldots, \pm e_d\}$ and $x \in \mathbb{Z}^d$. Then $\text{Tr}_e(x + \mathcal{C})^{-1}$ is a translation move in $[-l, l]^d$ of the cluster $x + e + \mathcal{C}$ in the direction $-e$. We may therefore always assume that the translation moves are chosen such that $\text{Tr}_e(x+\mathcal{C})^{-1} = \text{Tr}_{-e}(x+e+\mathcal{C})$.

Once we are able to move the mobile cluster around, we need to use it in order to move particles in its vicinity.

Definition 3.12 (Exchange move)**.** Fix a finite set $\mathcal{C} \subset \mathbb{Z}^d$, $l > 0$, $e \in \{\pm e_1, \ldots, \pm e_d\}$ and $x \in \mathbb{Z}^d$. An *exchange move* in $[-l, l]^d$ using the cluster $x + \mathcal{C}$ in the direction e is a T_{Ex}-step move $\text{Ex}_e(x + \mathcal{C})$ satisfying:

(1) $\operatorname{Dom} \text{Ex}_e(x + \mathcal{C}) = \{\eta \in \Omega : x + \mathcal{C} \text{ is empty}\}$
(2) $\text{Ex}_e(x+\mathcal{C})$ is a deterministic move, compatible with the permutation $(x+y, x+y+e)$, where $y = le$.
(3) For all $t \in \{0, \ldots, T - 1\}$, $x_t \in x + [-l, l]^d \cup \{x + (l + 1)e\}$ and $x_t + e_t \in x + [-l, l]^d \cup \{x + (l + 1)e\}$.

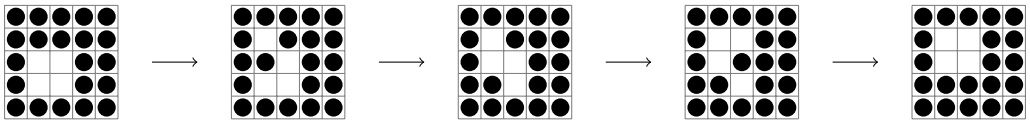

FIGURE 3.1. This is an illustration of the translation move in the model described in examples 2.2 and 3.15. The mobile cluster is given by an empty 2×2 square. In this figure we see how it could move one step up.

Definition 3.13 (Mobile cluster)**.** A *mobile cluster* $\mathcal{C}$ is a finite set of sites, for which there exists $l > 0$ such that $\mathrm{Tr}_e(x+\mathcal{C})$ and $\mathrm{Ex}_e(x+\mathcal{C})$ could be constructed for all e and x. Equivalently, by translation invariance, there exists $l > 0$ such that $\mathrm{Tr}_e(\mathcal{C})$ and $\mathrm{Ex}_e(\mathcal{C})$ could be constructed for all e.

A kinetically constrained lattice gas is called *noncooperative* if there exists a mobile cluster.

Example 3.14. The model in Example 2.1 is noncooperative—take $\mathcal{C} = \{1, 2\}$ and $l = 3$. We need to construct four moves: $\mathrm{Tr}_1(\mathcal{C}), \mathrm{Tr}_{-1}(\mathcal{C}), \mathrm{Ex}_1(\mathcal{C}), \mathrm{Ex}_{-1}(\mathcal{C})$.

$\mathrm{Tr}_1(\mathcal{C})$ will be a 2-step move $((\eta_0, \eta_1, \eta_2), (x_0, x_1), (e_0, e_1))$ operating on $\eta \in \mathrm{Dom}\,\mathrm{Tr}_1(\mathcal{C})$ as follows:

$$\eta_0 = \eta, \quad \eta_1 = \eta^{2,3} = (2,3)\eta, \quad \eta_2 = (2,3,1)\eta,$$
$$x_0 = 2, e_0 = 1, \quad x_1 = 1, e_1 = 1.$$

Recalling that $\eta \in \mathrm{Dom}\,\mathrm{Tr}_1(\mathcal{C})$ means $\eta(1) = \eta(2) = 0$, it is straightforward to verify that the move is well defined and that it is indeed a translation move. See Figure 1.1.

$\mathrm{Tr}_{-1}(\mathcal{C})$ is defined as $\mathrm{Tr}_1(-1+\mathcal{C})^{-1}$.

$\mathrm{Ex}_1(\mathcal{C})$ is the 1-step move exchanging the sites 3 and 4, which is allowed since 2 must be empty.

$\mathrm{Ex}_{-1}(\mathcal{C})$ could be constructed as the composition

$$\mathrm{Ex}_{-1}(\mathcal{C}) = \mathrm{Tr}_{-1}(-1+\mathcal{C})^{-1} \circ \mathrm{Tr}_{-1}(-2+\mathcal{C})^{-1} \circ \mathrm{Tr}_{-1}(-3+\mathcal{C})^{-1} \circ \mathrm{Tr}_{-1}(-4+\mathcal{C})^{-1} \circ \mathrm{Ex}_1(-5+\mathcal{C})$$
$$\circ\, \mathrm{Tr}_{-1}(-4+\mathcal{C}) \circ \mathrm{Tr}_{-1}(-3+\mathcal{C}) \circ \mathrm{Tr}_{-1}(-2+\mathcal{C}) \circ \mathrm{Tr}_{-1}(-1+\mathcal{C}) \circ \mathrm{Tr}_{-1}(\mathcal{C}).$$

The composition is well defined (recalling $\mathrm{Tr}_{-1}(x+\mathcal{C})^{-1} = \mathrm{Tr}_1(x-1+\mathcal{C})$, so its domain consists of the configurations where $x - 1 + \mathcal{C}$ is empty). Moreover, it is a composition of deterministic moves, and compatible with

$$(1,2,0)(0,1,-1)(-1,0,-2)(-2,-1,-3)(-3,-2,-4)(-2,-1)$$
$$(-4,-2,-3)(-3,-1,-2)(-2,0,-1)(-1,1,0)(0,2,1) = (-3,-4).$$

Example 3.15. The model in Example 2.2 is noncooperative, with $\mathcal{C} = \{e_1+e_2, e_1+2e_2, 2e_1+e_2, 2e_1+2e_2\}$ and $l = 3$. The construction of the multistep moves is the same as the previous example, see Figure 3.1.

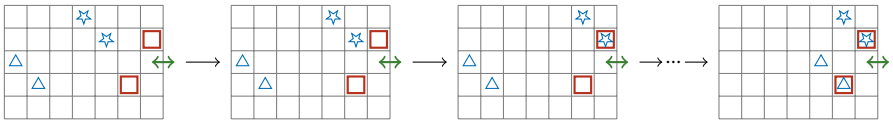

FIGURE 3.2. We see here how the exchange move could be constructed, see Claim 3.17. For the sake of this illustration, we assume that it suffices to empty the two sites marked with a red square in order to free the edge $(0, e_1)$ marked in green. The mobile cluster $\mathcal{C}$, marked with blue stars, is empty. In addition, a translation of $\mathcal{C}$, marked with blue triangles, is also empty. After applying the multistep move described in the figure the constraint is satisfied at the edge $(0, e_1)$, so we may exchange the two sites and move the mobile clusters back to their original position.

To conclude this section, we see in the following proposition that if we are able to construct, for any direction, a cluster that is free to move in that direction, then the model is noncooperative, i.e., there is some (possible very large) cluster that is able to move in all directions, and to exchange edges in its vicinity.

Proposition 3.16. *Assume that for any $e \in \{e_1, \ldots, e_d\}$ there exists $\mathcal{C}_e$ and l_e, such that $Tr_e(\mathcal{C}_e)$ exists. Then the model is noncooperative, i.e., there exists a mobile cluster $\mathcal{C}$.*

Proof. The construction of the cluster $\mathcal{C}$ is explained in the appendix of [25] (claims A11 and on). Since the result there is stated in a slightly different context (and with different notation), we explain here briefly how the cluster is constructed. The reader may consult [25] for any missing details.

Claim 3.17. Fix e. If $\mathrm{Tr}_e(\mathcal{C})$ exists for some $\mathcal{C}$ and l, then $\mathrm{Tr}_e(\mathcal{C}')$ and $\mathrm{Ex}_e(\mathcal{C}')$ exist for some $\mathcal{C}'$ and l'.

Proof. Without loss of generality $e = e_1$. Let $\{y_1, \ldots, y_k\} \in (\infty, 0] \times \mathbb{Z}^{d-1}$ be finite set of sites such that $c_{0,e} \geq 1$ if $\{y_1, \ldots, y_k\}$ is empty. This set has to exist since $\mathrm{Tr}_e(\mathcal{C})$ exists. Fix $\mathcal{C}' = \bigcup_{i=1}^{k} (y_i - ile_1 + \mathcal{C})$. Define $\mathrm{Ex}(\mathcal{C}')$ by translating the copies of $\mathcal{C}$ until $y_1, \ldots, y_k$ are all empty, then exchange 0 and e, and finally roll back the translation moves. See Figure 3.2. □

Claim 3.18. Assume $\mathrm{Tr}_1(\mathcal{C}_1)$, $\mathrm{Ex}_1(\mathcal{C}_1)$, $\mathrm{Tr}_2(\mathcal{C}_2)$, $\mathrm{Ex}_2(\mathcal{C}_2)$,...,$\mathrm{Tr}_k(\mathcal{C}_k)$, $\mathrm{Ex}_k(\mathcal{C}_k)$ are defined. Then there exist $\mathcal{C}'_k$ and l'_k such that for all $y \in [l_1, \infty]e_1 + \mathbb{Z}e_2 + \cdots + \mathbb{Z}e_k$ we may define a multistep move Ex_1^y exchanging $(y, y + e_1)$.

Proof. Consider first $k = 2$, and denote $\mathcal{C}_1 = \{x_1, \ldots, x_n\}$. We choose

$$\mathcal{C}'_2 = \mathcal{C}_1 \cup \left(\bigcup_{i=1}^{n} x_i - le_2 + \mathcal{C}_2 \right).$$

By applying translation and exchange moves using the cluster $x_i - le_2 + \mathcal{C}_2$, we are able to exchange x_i with $x_i + e_2$. Doing that for all i, we end up with an empty cluster $(e_2 + \mathcal{C}_1) \cup (\bigcup_{i=1}^{n} x_i - le_2 + \mathcal{C}_2)$. We can repeat the operation (with one additional translation move for

each i), reaching an empty cluster $(2e_2 + \mathcal{C}_1) \cup (\bigcup_{i=1}^n x_i - le_2 + \mathcal{C}_2)$. In fact, by adjusting the number of repetitions we are able to empty all sites of $(w + \mathcal{C}_1) \cup (\bigcup_{i=1}^n x_i - le_2 + \mathcal{C}_2)$ where $w = y - (y \cdot e_1)e_1$. Now, since $w + \mathcal{C}_1$ is empty, we can use $\mathrm{Tr}_1(w + \mathcal{C}_1)$ and $\mathrm{Ex}_1(w + \mathcal{C}_1)$ in order to exchange y and $y + e_1$. Rolling back all changes, we end up with the move Ex_1^w.

For larger values of k we follow the same construction by induction—use $\left|\mathcal{C}'_{k-1}\right|$ copies of $\mathcal{C}_k$ in order to move a single copy of $\mathcal{C}'_{k-1}$ in the e_k direction $y \cdot e_k$ times. Then apply (the translation of) $\mathrm{Ex}_1^{y - y\cdot e_k}$ in order to exchange y and $y + e_1$, and roll back to place $\mathcal{C}'_k$ in its original location. □

This claim allows us to define a cluster $\mathcal{C}'_d$, which allows exchanges in the direction e_1. We may construct in the same manner clusters allowing exchanges in any direction:

Corollary 3.19. *For any e, there exist $\bar{l}_e$ and $\overline{\mathcal{C}}_e$, such that we may define a multistep move $\mathrm{Ex}_e^y(x + \overline{\mathcal{C}}_e)$ exchanging $x + y$ with $x + y + e$ whenever $x + \overline{\mathcal{C}}_e$ is empty, for all y such that $y \cdot e \geq \bar{l}_e$.*

To conclude, consider $2d$ disjoint copies of the clusters defined in the corollary above placed on the diagonal—

$$\mathcal{C} = \left(\bigcup_{\alpha=1}^{d} -\alpha l(1, 1, \ldots, 1) + \overline{\mathcal{C}}_{e_\alpha}\right) \bigcup \left(\bigcup_{\alpha=1}^{d} \alpha l(1, 1, \ldots, 1) + \overline{\mathcal{C}}_{e_{-\alpha}}\right),$$

for large enough l to guarantee that the union is indeed disjoint. Now, in order to construct $\mathrm{Ex}_{e_\alpha}(\mathcal{C})$ we may simply use $\mathrm{Ex}_{e_\alpha}^y(x + \overline{\mathcal{C}}_{e_\alpha})$ with $x = -\alpha l(1, \ldots, 1)$ and $y = \alpha l e_\alpha - x$ (and analogously for $\mathrm{Ex}_{e_{-\alpha}}(\mathcal{C})$). In order to construct Tr_{e_α}, we first use the cluster $-\alpha l(1, \ldots, 1) + \overline{\mathcal{C}}_{e_\alpha}$ in order to move in the direction e_α all vacancies in $\bigcup_{\alpha=1}^d \left(\alpha l(1, 1, \ldots, 1) + \mathcal{C}_{e_{-\alpha}}\right)$. Then we use the cluster $\alpha l(1, \ldots, 1) + e_\alpha + \overline{\mathcal{C}}_{e_{-\alpha}}$ in order to move all vacancies in $\bigcup_{\alpha=1}^d \left(-\alpha l(1, 1, \ldots, 1) + \overline{\mathcal{C}}_{e_\alpha}\right)$ in the direction e_α. This concludes the proof of the proposition. □

4. Relaxation time on a finite box with a reservoir

In this section we consider noncooperative kinetically constrained lattice gases on a finite box $[L]^d$ with reservoirs on the boundary. In [2], the relaxation times of two models were studied, and a diffusive scaling was proven. We will follow their strategy, showing a diffusive scaling with power law dependence on q.

In order to define the relaxation time, we first write the Dirichlet form associated with the generator $\mathcal{L}_\mathrm{r}$ given in equation (2.2):

$$\mathcal{D}_\mathrm{r} f = \mu \left[\sum_{x \sim y \in \Lambda} c_{x,y} (\nabla_{x,y} f)^2\right] + \mu \left[\sum_{x \in \partial\Lambda} c_x (\nabla_x f)^2\right]. \tag{4.1}$$

Then the relaxation time is given by

$$\sup_{\substack{f:\Omega\to\mathbb{R} \\ \mathrm{Var}\, f \neq 0}} \frac{\mathrm{Var}\, f}{\mathcal{D}_\mathrm{r} f}. \tag{4.2}$$

The following theorem provides an upper bound on the relaxation time:

Theorem 4.1. *Consider a noncooperative kinetically constrained lattice gas on a finite box $\Lambda = [L]^d$ with reservoirs (see equation (2.2)) and empty boundary conditions. Fix a mobile cluster $\mathcal{C}$ of size N. Then for any $f : \Omega \to \mathbb{R}$,*

$$\operatorname{Var} f \leq C q^{-N-1} L^2 \mathcal{D}_r f,$$

where the variance is taken with respect to the equilibrium μ and $\mathcal{D}_r$ is the associated Dirichlet form given in equation (4.1).

4.1. **Proof.** We will first prove Theorem 4.1 when $q \leq \frac{1}{2}$, and then briefly explain how to adapt the proof for $q > \frac{1}{2}$.

We follow the steps of [2]—for any $x \in \Lambda$, we will define a multistep move that creates a mobile cluster at the boundary and uses it in order to flip the occupation at x. We will then prove the theorem using this multistep move together with the inequality

$$\operatorname{Var} f \leq q(1-q)\mu \left[\sum_{z \in \Lambda} (\nabla_z f)^2 \right]. \tag{4.3}$$

Lemma 4.2. *For any $z \in \Lambda$, there exists a T-step move $\mathit{Flip}_z = ((\eta_t), (x_t), (e_t))$ such that:*

(1) $\operatorname{Dom} \mathit{Flip}_z = \Omega_\Lambda$.
(2) For any η, the final configuration is given by $\eta_T(\eta) = \eta^z$.
(3) $T \leq CL$.
(4) The information loss $\operatorname{Loss} \mathit{Flip}_z \leq C$.
(5) The energy barrier $\operatorname{EB} \mathit{Flip}_z \leq N + 1$.
(6) For any $t \in \{0, \ldots, T\}$, $x(t) \in z + \Delta$, where $\Delta \subset \mathbb{Z}^d$ is fixed and $|\Delta| \leq CL$.
(7) Each site $x \in \Lambda$ is changed a bounded number of times, i.e., $\{t : x_t = x\} \leq C$.

Proof. Let $\overline{z} = z - e_1 \cdot z$, and consider the configuration $\overline{\eta}$ defined on the infinite lattice as follows

$$\overline{\eta}(y) = \begin{cases} \eta(y) & \text{if } y \in \Lambda, \\ 1 - \eta(z) & \text{if } y = \overline{z}, \\ 0 & \text{if } y \in \overline{z} - l e_1 + \mathcal{C}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.4}$$

We will define a $\overline{T}$-step move $\overline{M}$ operating on this configuration by composing exchange and translation moves as follows—

(1) Using the mobile cluster $\overline{z} - l e_1 + \mathcal{C}$, apply the exchange move $\mathrm{Ex}_1(\overline{z} - l e_1 + \mathcal{C})$ (Definition (3.12)) in order to exchange $\overline{z}$ with $\overline{z} + e_1$.
(2) Apply the translation move $\mathrm{Tr}_1(\overline{z} - l e_1 + \mathcal{C})$ (Definition (3.10)) in order to move the cluster $\overline{z} - l e_1 + \mathcal{C}$ one step to the right.

(3) Continue to apply these two moves alternatingly until reaching x, i.e.,

$$\mathrm{Tr}_1(y_k+\mathcal{C})\circ\mathrm{Ex}_1(y_k+\mathcal{C})\circ\cdots\circ\mathrm{Tr}_1(y_1+\mathcal{C})\circ\mathrm{Ex}_1(y_1+\mathcal{C})\circ\mathrm{Tr}_1(y_0+\mathcal{C})\circ\mathrm{Ex}_1(y_0+\mathcal{C}),$$

where $y_i=\overline{z}-le_1+ie_1$ for all i, and k is chosen such that $y_k=z-2e_1$.

(4) Apply the exchange move $\mathrm{Ex}_1(y_k+e_1+\mathcal{C})$ in order to exchange y_k+e_1 with z.

(5) Wind back the exchanges and translations of step 3 and move the mobile cluster back to $\overline{z}-le_1+\mathcal{C}$.

Putting everything together, we obtain

$$\begin{aligned}\overline{M}=\mathrm{Ex}_1(y_0+\mathcal{C})\circ\mathrm{Tr}_{-1}(y_1+\mathcal{C})\circ\cdots\circ\mathrm{Ex}_1(y_k+\mathcal{C})\circ\mathrm{Tr}_{-1}(y_{k+1}+\mathcal{C})\circ\mathrm{Ex}_1(y_{k+1}+\mathcal{C})\\ \circ\mathrm{Tr}_1(y_k+\mathcal{C})\circ\mathrm{Ex}_1(y_k+\mathcal{C})\circ\cdots\circ\mathrm{Tr}_1(y_0+\mathcal{C})\circ\mathrm{Ex}_1(y_0+\mathcal{C}).\end{aligned}$$

We have thus constructed a multistep move $\overline{M}$ with the following properties:

(1) $\overline{\eta}\in\mathrm{Dom}\,\overline{M}$ for any $\eta\in\Omega_\Lambda$.
(2) $\overline{M}$ is compatible with the transposition exchanging $\overline{z}$ and z.
(3) $\overline{T}\le CL$.
(4) $\mathrm{Loss}\,\overline{M}=0$ and $\mathrm{EB}\,\overline{M}=0$.
(5) All exchanges occur in a tube $\overline{z}+[-l,L]\times[-l,l]^{d-1}$ for some (large enough) fixed l.

The move Flip_z that we construct will simply be the restriction of $\overline{M}$ to Λ—if we denote $\overline{M}=(\overline{\eta}_t,\overline{x}_t,\overline{e}_t)$, then Flip_z will be such that, for any $y\in\Lambda$,

$$\eta_t(y)=\overline{\eta}_t(y).$$

All that is left is to verify that this move satisfies the required properties:

(1) It is well-defined on the entire Ω_Λ—for any $\eta\in\Omega_\Lambda$ we know that $\overline{\eta}$ defined above is in $\mathrm{Dom}\,\overline{M}$. In addition, a transition in $\overline{M}$ outside Λ does not change η_t, a transition on the boundary corresponds to a reservoir term for η_t, and a transition inside Λ which is allowed for $\overline{\eta}_t$ is certainly allowed for η_t. This means that all transitions in Flip_x are allowed, making it a valid move.
(2) Since $\overline{z}\notin\Lambda$ and $\overline{\eta}(\overline{z})=1-\eta(z)$, the fact that $\overline{M}$ is compatible with the transposition exchanging $\overline{z}$ and z implies that the final configuration of Flip_z is η^z.
(3) $T=\overline{T}\le CL$.
(4) In order to reconstruct $\overline{\eta}_t$ from η_t it is enough to know the occupation at some finite box to the left of $\overline{z}$. Since $\overline{M}$ has 0 loss of information, the size of this box bounds the loss of information.
(5) The number of vacancies in η_t is certainly smaller than that of $\overline{\eta}_t$, which exceeds the number of vacancies of η by at most $N+1$.
(6) Choosing $\Delta=\overline{z}+[-l-L,L]\times[-l,l]^{d-1}$ will suffice.

(7) Since the exchange and translation moves operate locally, a site z could be "touched" by a bounded number of such moves, each of which being able to change z a bounded number of times. □

We will now use Lemma 4.2 in order to prove Theorem 4.1. Start by considering, for each $z \in \Lambda$, the T-step move $\text{Flip}_z = (\eta^z, x^z, e^z)$, and using it in order to write

$$\begin{aligned}(\nabla_z f)^2 &= \left(\sum_{t=0}^{T-1} \nabla_t f(\eta_t^z)\right)^2 \\ &\leq CL \sum_{t=0}^{T-1} \left(\nabla_t f(\eta_t^z)\right)^2,\end{aligned}$$

where ∇_t stands for $\nabla_{x_t^z, x_t^z + e_t^z}$ for a bulk exchange ($\eta_{t+1} = \eta_t^{x_t^z, x_t^z + e_t^z}$), or $\nabla_{x_t^z}$ for a boundary flip ($\eta_{t+1} = \eta_t^{x_t^z}$).

Then by equation 4.3

$$\begin{aligned}&\operatorname{Var} f \leq CLq(1-q)\mu\left[\sum_{z\in\Lambda}\sum_{t=0}^{T-1}\left(\nabla_t f(\eta_t^z)\right)^2\right] \\ &= CLq\sum_{\eta\in\Omega_\Lambda}\mu(\eta)\sum_{z\in\Lambda}\sum_{t}\sum_{\eta'\in\Omega_\Lambda}\sum_{x\in z+\Delta}\sum_{e}\mathbb{1}_{\text{bulk exchange}}\mathbb{1}_{x_t^z(\eta)=x}\mathbb{1}_{e_t^z(\eta)=e}\mathbb{1}_{\eta_t(\eta)=\eta'}\, c_{x,x+e}(\eta')\left(\nabla_{x,x+e}f(\eta')\right)^2 \\ &\quad + CLq\sum_{\eta\in\Omega_\Lambda}\mu(\eta)\sum_{z\in\Lambda}\sum_{t}\sum_{\eta'\in\Omega_\Lambda}\sum_{x\in\partial\Lambda\cap(z+\Delta)}\mathbb{1}_{\text{bounday flip}}\mathbb{1}_{x_t^z(\eta)=x}\mathbb{1}_{\eta_t=\eta'}\left(\nabla_x f(\eta')\right)^2 \\ &\leq CLq\sum_{x\in\Lambda}\sum_{e}\sum_{\eta'\in\Omega_\Lambda}\mu(\eta')c_{x,x+e}(\eta')\left(\nabla_{x,x+e}f(\eta')\right)^2\sum_{\eta\in\Omega_\Lambda}\frac{\mu(\eta)}{\mu(\eta')}\sum_{z\in x-\Delta}\sum_{t}\mathbb{1}_{x_t^z(\eta)=x}\mathbb{1}_{\eta_t(\eta)=\eta'} \\ &\quad + CL\sum_{x\in\partial\Lambda}\sum_{\eta'\in\Omega_\Lambda}\mu(\eta')c_x(\eta')\left(\nabla_x f(\eta')\right)^2\sum_{\eta\in\Omega_\Lambda}\frac{\mu(\eta)}{\mu(\eta')}\sum_{z\in x-\Delta}\sum_{t}\mathbb{1}_{x_t^z(\eta)=x}\mathbb{1}_{\eta_t=\eta'}.\end{aligned}$$

We will now use the properties of Flip_z in order to bound the different terms above. First, since we assume $q \leq \frac{1}{2}$,

$$\frac{\mu(\eta)}{\mu(\eta')} \leq q^{-\operatorname{EB}(\text{Flip}_z)} = q^{-N-1}.$$

The bound on the loss of information allows us to write $\sum_{\eta\in\Omega_\Lambda}\mathbb{1}_{\eta_t(\eta)=\eta'} \leq C$.

The last property of the flip move implies that $\sum_{t=0}^{T}\mathbb{1}_{x_t^z(\eta)=x} \leq C$.

Putting everything together, we obtain

$$\begin{aligned}\operatorname{Var} f &\leq CLq^{-N-1}\left|\Delta\right|\sum_{x\in\Lambda}\sum_{e}\sum_{\eta'\in\Omega_\Lambda}\mu(\eta')c_{x,x+e}(\eta')\left(\nabla_{x,x+e}f(\eta')\right)^2 \\ &\quad + CLq^{-N-1}\left|\Delta\right|\sum_{x\in\partial\Lambda}\sum_{\eta'\in\Omega_\Lambda}\mu(\eta')c_x(\eta')\left(\nabla_x f(\eta')\right)^2 \\ &\leq CL^2q^{-N-1}\mathcal{D}_\Lambda f.\end{aligned}$$

This concludes the proof when $q \leq \frac{1}{2}$.

The case $q > \frac{1}{2}$ could be thought of as a negative temperature setting, so the relevant quantity is the *negative* energy barrier—rather than counting the excess vacancies, we should count the excess particles. By changing the definition of $\overline{\eta}$ given in equation (4.4) such that $\overline{\eta}(y) = 0$ if $y \notin \Lambda \cup \{\overline{z}\}$, we can construct the Flip_z in the same manner, such that at each t the number of particles in η_t exceeds the number of particles in η by at most 1. The only estimate that changes is that of $\frac{\mu(\eta)}{\mu(\eta')}$, which becomes $\frac{\mu(\eta)}{\mu(\eta')} \leq (1-q)^{-1}$, and the rest of the proof follows. □

5. Relaxation time in a closed system

In this section we consider models on a finite box $\Lambda = [L]^d$, with no reservoirs. In this setting the total number of particles is fixed, hence μ cannot be ergodic. Moreover, even if we condition μ to some fixed number of vacancies k, the measure that we obtain is in general not ergodic due to the constraint.

In particular, at least if q is not too large, one may construct *blocked configuration*. These are configurations where no particle is allowed to jump, and therefore do not change during the dynamics (see, e.g., Figure 1.2). If $k < \left(\frac{L}{R+1}\right)^d$ (where R is the range of the constraint), we may place the vacancies such that no two empty sites are at distance less than R. Since the model is nondegenerate the constraint is not satisfied for the edges adjacent to a vacancy, and the configuration is indeed blocked.

For noncooperative models, we note that two configurations containing a mobile cluster, at least for k large enough, are always in the same ergodic component—consider two configurations η and η' with k vacancies, each containing a mobile cluster, $x + \mathcal{C}$ and $x' + \mathcal{C}'$ respectively. Assuming $k > |\mathcal{C}| + |\mathcal{C}'|$, we may use the translation and exchange moves on η with the cluster $x + \mathcal{C}$ in order to move vacancies to $x' + \mathcal{C}'$. Then we use the translation and exchange moves with the cluster $x' + \mathcal{C}'$ to move around all other vacancies to their locations in η'.

We therefore define the ergodic configurations as follows:

Definition 5.1. Consider a family of mobile clusters $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$. The set of ergodic configurations with k vacancies, denoted Ω_k, is given by all configurations η containing k vacancies connected to a configuration that contains an empty translation of a mobile cluster. More precisely, $\eta \in \Omega_k$ if it contains k vacancies, and there exists a T-step move $M = ((\eta_t), (x_t), (e_t))$, a site $x \in \Lambda$, and some $i \in [m]$, such that $\eta \in \text{Dom}\, M$ and all sites of $x + \mathcal{C}_i$ are empty for the configuration $\eta_T(\eta)$.

The *equilibrium measure* μ_k is the uniform measure on Ω_k. We denote in this section $\mu = \mu_k$.

The discussion above implies the following fact:

Fact 5.2. *For any family of mobile clusters $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$, and any $k > 2 \max_{i=1}^m |\mathcal{C}_i|$, the measure μ is ergodic.*

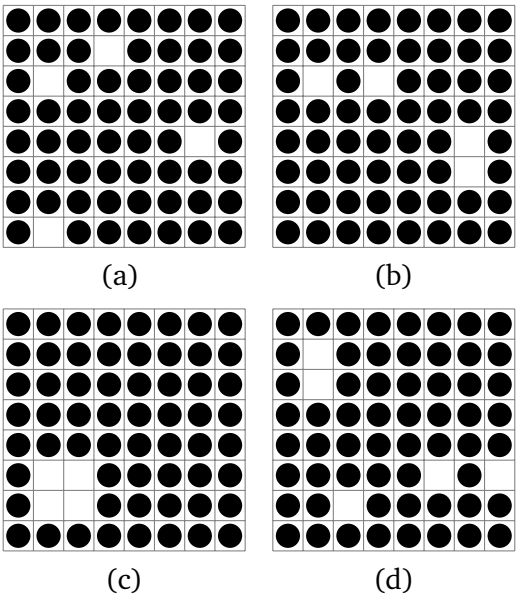

FIGURE 5.1. A few configurations in the model of Example 2.2 defined on a finite box. The mobile cluster of this model is a 2×2 square (see Example 3.15 and Figure 3.1). Configuration (a) is blocked hence not ergodic, configuration (b) is not blocked but still not ergodic, configuration (c) contains a mobile cluster hence ergodic, and configuration (d) is ergodic even though no small region contains a mobile cluster. See Example 5.4.

Example 5.3. Consider the model of Example 2.1, and the family of mobile clusters $\{\{1,2\}, \{1,3\}\}$. If a configuration η does not contain an empty translation of either cluster, it is blocked, since all allowed transitions for the dynamics involve two vacancies at distance at most 2. Therefore, the ergodic configurations in this models are those containing an empty translation of $\{1,3\}$ or $\{1,2\}$.

Example 5.4. In the model introduced in Example 2.2 the ergodic component is more complicated. One can find configurations that are not blocked but still not ergodic, or configurations which are ergodic but do not contain a mobile cluster of size smaller than L. An explicit description of Ω_k for this model seems to be much more difficult to find than the 1 dimensional case. See Figure 5.1.

In view of these examples, we will restrict our discussion to models with easily identifiable set of ergodic configurations:

Hypothesis 5.5. There exists a finite family of mobile clusters, $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$, such that

$$\Omega_k = \{\eta : \text{there exist } x \in \Lambda \text{ and } i \in [m] \text{ for which } x + \mathcal{C}_i \text{ is in } \Lambda \text{ and empty}\}\,.$$

Fix $k > \max_{i=1}^{m} |\mathcal{C}_i|$, so Ω_k is nonempty and the measure μ is well defined. The Dirichlet form associated with the generator (2.1) and the (reversible) measure μ is given by

$$\mathcal{D}f = \mu \left[\sum_{\substack{x,y\in\Lambda \\ x\sim y}} c_{x,y}(\nabla_{x,y} f)^2 \right]. \tag{5.1}$$

The result of this section is a bound on the relaxation time of 1 dimensional models satisfying Hypothesis 5.5:

Theorem 5.6. *Consider a noncooperative kinetically constrained lattice gas with occupied boundary conditions in one dimension satisfying Hypothesis 5.5, and let $k = \lfloor qL \rfloor$ for some $q \in (0,1)$. Then for L large enough and any $f : \Omega_k \to \mathbb{R}$*

$$\operatorname{Var} f \leq C q^C L^2 \mathcal{D} f,$$

where the variance is taken with respect to $\mu = \mu_k$ and $\mathcal{D}$ is the associated Dirichlet form given in equation (5.1).

5.1. **Proof.** The overall scheme of the proof is similar to that of [11]—we first create many mobile clusters, and then use them in order to exchange the occupation of pairs of sites. This will allow us to compare our model with the simple exclusion process on the complete graph. The main difference between the proof here and the one presented in [11] is that the creation of the mobile clusters is accomplished without resorting to a perturbed model.

We start with a few definitions, which will depend on a fixed arbitrary mobile cluster $\mathcal{C}$ of size N, and an integer $\lambda > \frac{2N}{q}$ such that $\mathcal{C}_i \subset [\lambda]$ for all $i \in \{1, \dots, k\}$.

Definition 5.7. A *box* (of size λ) is a subset of Λ of the type $\lambda i + [\lambda]$, for $i \in \mathbb{Z}$. We may assume that $\frac{L}{\lambda} \in \mathbb{N}$ by the same monotonicity argument as in [22, Remark 3.1], and denote the set of boxes

$$\mathcal{B} = \{\lambda i + [\lambda],\, i \in \mathbb{Z} \cap [0, L/\lambda - 1]\}.$$

Definition 5.8. A *good box* is a box containing an empty translation of $\mathcal{C}$.

A *pregood box* is a box containing at least N vacancies (recall $N = |\mathcal{C}|$).

We denote by G the event that at least $k_0 = \left\lfloor \lambda^{-N} \left(\frac{k}{4\lambda} - 1 \right) \right\rfloor$ boxes are good. We assume L (and therefore k) large enough so that $k_0 > 0$.

Claim 5.9. For any $\eta \in \Omega_k$, at least $\frac{k}{2\lambda}$ boxes are pregood.

Proof. Let n_v be the number of boxes containing exactly v vacancies, so the number of pregood boxes is $\sum_{v=N}^{\lambda} n_v$. Then

$$k = \sum_{v=0}^{\lambda} v n_v = \sum_{v=0}^{N-1} v n_v + \sum_{v=N}^{\lambda} v n_v$$

$$\leq N\,|\mathcal{B}| + \lambda \sum_{v=N}^{\lambda} n_v \leq \frac{k}{2} + \lambda \cdot \#\text{pregood boxes.} \qquad \square$$

Definition 5.10. Let Σ be the set whose elements are of the type $s = (o, \sigma)$, for $o \in \{+, -\}$ and $\sigma = (\sigma_B)_{B \in \mathcal{B}}$, where σ_B is a permutation of the sites of B for any box $B \in \mathcal{B}$.

For a configuration $\eta \in \Omega_k$ and $s \in \Sigma$, we construct the configuration $s\eta$ as follows:

(1) Find the the first mobile cluster in the orientation o, that is, the site $z \in \Lambda$ together with $i \in \{1, \ldots, k\}$ such that:
   (a) $z + \mathcal{C}_i$ is empty for some $i \in \{1, \ldots, k\}$.
   (b) z is the leftmost site satisfying (a) if $o = +$, and the rightmost if $o = -$. Differently stated, for any $y \neq z$ such that $y + \mathcal{C}_j$ is empty for some $j \in \{1, \ldots, k\}$, $oz < oy$.

(2) Identify the set $\mathcal{B}_o$ of boxes after z, that is, the boxes $B \in \mathcal{B}$ in which all sites are strictly to the right of $z + \mathcal{C}_i$ if $o = +$, or strictly to its left in the case $o = -$.

(3) For $x \in \Lambda$, denoting by B the box containing x,

$$s\eta(x) = \begin{cases} \eta(x) & \text{if } B \notin \mathcal{B}_o, \\ \eta(\sigma_B^{-1} x) & \text{if } B \in \mathcal{B}_o. \end{cases}$$

Observation 5.11. The action defined above is bijective—for any $s \in \Sigma$ we can define $s^{-1} \in \Sigma$ by inverting each permutation and keeping the orientation fixed. Then $ss^{-1}\eta = \eta$ for any $\eta \in \Omega_k$.

Claim 5.12. Fix $\eta \in \Omega_k$. Then

$$\frac{|\{s \in \Sigma : s\eta \in G\}|}{|\Sigma|} \geq \frac{1}{4}.$$

Proof. We use the notation of Definition 5.10. $\cup_{o \in \{\pm\}} \mathcal{B}_o$ contains all boxes, except for a maximum of 2 boxes containing sites of the mobile cluster. By Claim 5.9, at least $\frac{k}{2\lambda} - 2$ of them are pregood. Hence, there is an orientation $o^\star \in \{+, -\}$, such that the number of pregood boxes in $\mathcal{B}_{o^\star}$ is at least $\frac{k/2\lambda - 2}{2}$.

Let $s = (o, \sigma)$ be an element of Σ chosen uniformly at random. Equivalently, we can say that o is chosen uniformly at random from $\{+, -\}$ and each permutation in σ is chosen uniformly at random, all independently of one another. As we have seen above, under this measure, denoting by p the number of boxes in $\mathcal{B}_o$ that are pregood for η,

$$\mathbb{P}\left[p \geq \frac{k}{4\lambda} - 1\right] \geq \mathbb{P}[o = o^\star] = \frac{1}{2}.$$

For each box $B \in \mathcal{B}_o$ which is pregood for η, the probability that B is good for $s\eta$ is at least λ^{-N}. Hence, conditioning on $p \geq \frac{k}{4l} - 1$, the number of good boxes for $s\eta$ is dominating a binomial random variable of parameters $\frac{k}{4l} - 1$ and λ^{-N}. The median of the latter is

$\lambda^{-N}\left(\frac{k}{4\lambda}-1\right)=k_0$, hence

$$\mathbb{P}\left[\#\text{good boxes for } s\eta \geq k_0 | p \geq \frac{k}{4\lambda}-1\right] \geq \frac{1}{2}.$$

This concludes the proof. □

In order to bound the variance of f, we start by writing

$$\begin{aligned}
\operatorname{Var} f &= \frac{1}{2}\sum_{\eta,\eta'\in\Omega_k}\mu(\eta)\mu(\eta')\left(f(\eta)-f(\eta')\right)^2 \\
&= \frac{1}{2}\sum_{\eta,\eta'\in\Omega_k}\mu(\eta)\mu(\eta')\frac{1}{|\{s\in\Sigma : s\eta\in G\}|^2}\sum_{s\in\Sigma}\mathbb{1}_{s\eta\in G}\sum_{s'\in\Sigma}\mathbb{1}_{s'\eta'\in G}\left(f(\eta)-f(\eta')\right)^2 \\
&\leq \frac{C}{|\Sigma|^2}\sum_{s,s'\in\Sigma}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\mathbb{1}_{s\eta\in G}\mathbb{1}_{s'\eta'\in G}\left(f(\eta)-f(\eta')\right)^2 \\
&= \frac{C}{|\Sigma|^2}\sum_{s,s'\in\Sigma}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\mathbb{1}_{s\eta\in G}\mathbb{1}_{s'\eta'\in G}\left(f(\eta)-f(s\eta)+f(s\eta)-f(s'\eta')+f(s'\eta')-f(\eta')\right)^2 \\
&\leq \frac{C}{|\Sigma|^2}\sum_{s,s'\in\Sigma}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\mathbb{1}_{s\eta\in G}\mathbb{1}_{s'\eta'\in G}\left(f(\eta)-f(s\eta)\right)^2 \\
&\quad +\frac{C}{|\Sigma|^2}\sum_{s,s'\in\Sigma}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\mathbb{1}_{s\eta\in G}\mathbb{1}_{s'\eta'\in G}\left(f(s\eta)-f(s'\eta')\right)^2 \\
&\quad +\frac{C}{|\Sigma|^2}\sum_{s,s'\in\Sigma}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\mathbb{1}_{s\eta\in G}\mathbb{1}_{s'\eta'\in G}\left(f(s'\eta')-f(\eta')\right)^2 \\
&\leq \frac{C}{|\Sigma|}\sum_{s}\sum_{\eta}\mu(\eta)\left(f(\eta)-f(s\eta)\right)^2+\frac{C}{|\Sigma|^2}\sum_{s,s'\in\Sigma}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\mathbb{1}_{s\eta\in G}\mathbb{1}_{s'\eta'\in G}\left(f(s\eta)-f(s'\eta')\right)^2 \\
&= \mathrm{I}+\mathrm{II}.
\end{aligned}$$

In order to finish the proof of the theorem, it is left to show that

$$\mathrm{I} \leq Cq^{-C}L^2\mathcal{D}_\Lambda f, \tag{5.2}$$

$$\mathrm{II} \leq Cq^{-C}L^2\mathcal{D}_\Lambda f. \tag{5.3}$$

Let us start with inequality (5.2).

Claim 5.13. For any $s=(o,\sigma)\in\Sigma$ and $z\in\Lambda$ there exists a T-step move $M_{s,z}=((\eta_t),(x_t),(e_t))$ satisfying:

(1) $\operatorname{Dom} M_s=\{\eta\in\Omega_k : z \text{ is the first mobile cluster in } \eta \text{ for the orientation } o\}$.
(2) $\eta_T(\eta)=s\eta$ for any $\eta\in\operatorname{Dom} M_s$.
(3) $T\leq Cl^3L$.
(4) $\operatorname{Loss} M_s=0$.
(5) Each site $x\in\Lambda$ is exchanged at most $C\lambda^3$ times. Moreover,

$$|\{t \text{ such that } x_t(\eta)=x \text{ for some } \eta\in\operatorname{Dom} M_{s,z}\}|\leq C\lambda^3.$$

Proof. Assume for simplicity $o = +$, the case $o = -$ is analogous.

We start with the mobile cluster at z, and use the translation move (Definition 3.10) $L-\lambda-z$ times in order to move it to the box $[L - 2\lambda + 1, L - \lambda]$. The permutation $\sigma_{[L-\lambda+1,L]}$ can be decomposed as a product of at most $C\lambda^2$ nearest neighbor transpositions (see, e.g., [18, Section 5.2.2]). We apply them one by one, where at each step in order to exchange $L-\lambda+x$ with $L - \lambda + x + 1$ we move the cluster x times to the right using the translation move (Definition 3.10), then exchange $L - \lambda + x$ with $L - \lambda + x + 1$ using the exchange move (Definition 3.12), and finally move the cluster x times to the left. Each transposition takes $2xT_{\text{Tr}} + T_{\text{Ex}} < Cl$ steps.

Once the permutation $\sigma_{[L-\lambda+1,L]}$ has been applied, we move the cluster λ steps to the left, to the box $[L - 3\lambda + 1, L - 2\lambda]$, and apply as before the permutation $\sigma_{[L-2\lambda+1,L-\lambda]}$ to the box $[L - 2\lambda + 1, L - \lambda]$. Continue in the same manner until all boxes in $\mathcal{B}_+$ are rearranged, and move the cluster back to z.

The verification of 2-5 is immediate. □

We now use the move $M_{s,z} = ((\eta_t^{s,z}), (x_t^{s,z}), (e_t^{s,z}))$ in order to bound the term I: for any $s \in \Sigma$,

$$\begin{aligned}
\sum_\eta \mu(\eta) \left(f(\eta) - f(s\eta)\right)^2 &= \sum_\eta \mu(\eta) \sum_{z\in\Lambda} \mathbb{1}_{\eta\in \mathrm{Dom}\, M_{s,z}} \left(f(\eta) - f(s\eta)\right)^2 \\
&= \sum_\eta \mu(\eta) \sum_{z\in\Lambda} \mathbb{1}_{\eta\in \mathrm{Dom}\, M_{s,z}} \left(\sum_{t=0}^{T-1} \nabla_{x_t^{s,z}, x_t^{s,z}+e_t^{s,z}} f(\eta_t^{s,z})\right)^2 \\
&\le C \sum_\eta \mu(\eta) \sum_{z\in\Lambda} T \sum_{\eta'\in\Omega} \sum_{x\in\Lambda} \mathbb{1}_{\eta\in \mathrm{Dom}\, M_{s,z}} \sum_{t=0}^{T-1} \mathbb{1}_{\eta'=\eta_t^{s,z}} \mathbb{1}_{x=x_t^{s,z}} c_{x,x+1}(\eta') \left(\nabla_{x,x+1} f(\eta')\right)^2 \\
&\le C\lambda^6 L^2 \sum_{\eta'} \mu(\eta') \sum_{x\in\Lambda} c_{x,x+1}(\eta') \left(\nabla_{x,x+1} f(\eta')\right)^2 = C\lambda^6 L^2 \mathcal{D} f.
\end{aligned}$$

Therefore

$$\frac{C}{|\Sigma|} \sum_s \sum_\eta \mu(\eta) \left(f(\eta) - f(s\eta)\right)^2 \le C\lambda^6 L^2 \mathcal{D} f.$$

For q small we may choose $\lambda < \frac{2N+1}{q}$ and inequality (5.2) is satisfied. For q large the q and λ dependence could be put it the constant C, proving inequality (5.2) for all q.

We move to inequality (5.3). Start by noting that, thanks to the bijectivity of s and s', we can change variables in the sum to obtain

$$\begin{aligned}
\text{II} &= \frac{C}{|\Sigma|^2} \sum_{s,s'\in\Sigma} \sum_{\eta,\eta'} \mu(\eta)\mu(\eta') \mathbb{1}_{\eta\in G} \mathbb{1}_{\eta'\in G} \left(f(\eta) - f(\eta')\right)^2 \\
&= C \sum_{\eta,\eta'} \mu(\eta)\mu(\eta') \mathbb{1}_{\eta\in G} \mathbb{1}_{\eta'\in G} \left(f(\eta) - f(\eta')\right)^2.
\end{aligned}$$

Since under the good event there are at least k_0 sites x for which $x+\mathcal{C}$ is empty,

$$\begin{aligned}\text{II} &\le C\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\mathbb{1}_{\eta\in G}\mathbb{1}_{\eta'\in G}\,\frac{1}{k_0}\sum_{z\in\Lambda}\mathbb{1}_{z+\mathcal{C}\text{ is empty for }\eta}\frac{1}{k_0}\sum_{z'\in\Lambda}\mathbb{1}_{z'+\mathcal{C}\text{ is empty for }\eta'}\left(f(\eta)-f(\eta')\right)^2\\ &\le \frac{C}{k_0^2}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\sum_{z,z'}\mathbb{1}_{z+\mathcal{C}\text{ is empty for }\eta}\mathbb{1}_{z'+\mathcal{C}\text{ is empty for }\eta'}\left(f(\eta)-f(\eta')\right)^2.\end{aligned}$$

For η such that $z+\mathcal{C}$ is empty, let $\Theta_z\eta$ be the outcome of z translations moves to the left. That is, Θ_z is the permutation compatible with $\mathrm{Tr}_{-1}(1+\mathcal{C})\circ\cdots\circ\mathrm{Tr}_{-1}(z+\mathcal{C})$. We can then write II as

$$\begin{aligned}\text{II} &\le \frac{C}{k_0^2}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\sum_{z,z'}\mathbb{1}_{\eta(z+\mathcal{C})=0}\mathbb{1}_{\eta'(z'+\mathcal{C})=0}\left(f(\eta)-f(\Theta_z\eta)+f(\Theta_z\eta)-f(\Theta_{z'}\eta')+f(\Theta_{z'}\eta')-f(\eta')\right)^2\\ &\le \frac{C}{k_0^2}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\sum_{z,z'}\mathbb{1}_{\eta(z+\mathcal{C})=0}\mathbb{1}_{\eta'(z'+\mathcal{C})=0}\left(f(\eta)-f(\Theta_z\eta)\right)^2\\ &\qquad+\frac{C}{k_0^2}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\sum_{z,z'}\mathbb{1}_{\eta(z+\mathcal{C})=0}\mathbb{1}_{\eta'(z'+\mathcal{C})=0}\left(f(\Theta_z\eta)-f(\Theta_{z'}\eta')\right)^2\\ &\qquad+\frac{C}{k_0^2}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\sum_{z,z'}\mathbb{1}_{\eta(z+\mathcal{C})=0}\mathbb{1}_{\eta'(z'+\mathcal{C})=0}\left(f(\Theta_{z'}\eta')-f(\eta')\right)^2.\\ &\le \frac{CL}{k_0^2}\sum_{\eta}\mu(\eta)\sum_{z}\mathbb{1}_{\eta(z+\mathcal{C})=0}\left(f(\eta)-f(\Theta_z\eta)\right)^2\\ &\qquad+\frac{CL^2}{k_0^2}\sum_{\eta,\eta'}\mu(\eta)\mu(\eta')\mathbb{1}_{\eta(\mathcal{C})=0}\mathbb{1}_{\eta'(\mathcal{C})=0}\left(f(\eta)-f(\eta')\right)^2\\ &= \text{III}+\text{IV}.\end{aligned}$$

The term III could be bounded using the T-step move $M=((\eta_t),(x_t),(e_t))$ resulted from the composition of z translations to the left—it is not difficult to see that $T\le CL$, that it has 0 loss, and that each edge is flipped a bounded number of times. Therefore

$$\begin{aligned}\text{III} &\le \frac{CL^2}{k_0^2}\sum_{\eta}\mu(\eta)\sum_{z}\mathbb{1}_{\eta(z+\mathcal{C})=0}\sum_{\eta'}\sum_{x}\sum_{t=0}^{T-1}\mathbb{1}_{\eta'=\eta_t}\mathbb{1}_{x_t=x}c_{x,x+1}(\eta')\left(\nabla_{x,x+1}f(\eta')\right)^2\\ &\le \frac{CL^3}{k_0^2}\sum_{\eta'}\mu(\eta')c_{x,x+1}(\eta')\sum_{x}\left(\nabla_{x,x+1}f(\eta')\right)^2\le\frac{CL^2}{k_0^2}\,L\mathcal{D}f\le Cq^{-C}L\,\mathcal{D}f.\end{aligned}$$

In order to estimate the last term IV, we need two ingredients—first, let $\overline{\Omega}_{k-N}$ be the space of configurations on $\Lambda\setminus\mathcal{C}$ with $k-N$ particles, endowed with the uniform measure $\overline{\mu}$. Note that to any configuration $\eta\in\Omega_k$ in which $\mathcal{C}$ is empty we can associate a configuration $\overline{\eta}\in\overline{\Omega}_{k-N}$ and vice versa. We may also define the function $\overline{f}:\overline{\Omega}\to\mathbb{R}$, given by $\overline{f}(\overline{\eta})=f(\eta)$. Then

$$\text{IV}=\frac{CL^2}{k_0^2}\frac{\left|\overline{\Omega}_{k-N}\right|^2}{\left|\Omega_k\right|^2}\sum_{\overline{\eta},\overline{\eta}'}\overline{\mu}(\overline{\eta})\overline{\mu}(\overline{\eta}')\left(\overline{f}(\overline{\eta})-\overline{f}(\overline{\eta}')\right)^2.$$

Note that the variance of $\overline{f}$ with respect to the measure $\overline{\mu}$ is given by

$$\mathrm{Var}_{\overline{\mu}}\,\overline{f} = \frac{1}{2}\sum_{\overline{\eta},\overline{\eta}'}\overline{\mu}(\overline{\eta})\overline{\mu}(\overline{\eta}')\left(\overline{f}(\overline{\eta})-\overline{f}(\overline{\eta}')\right)^2.$$

We can therefore bound IV using the relaxation time of the simple exclusion process on the complete graph [7, 8], expressed in the following Poincaré inequality:

$$\mathrm{Var}_{\overline{\mu}}\,\overline{f} \le \frac{1}{L-N}\sum_{\overline{\eta}}\overline{\mu}(\overline{\eta})\sum_{y,z\in\Lambda\setminus\mathcal{C}}\left(\nabla_{x,y}\overline{f}(\overline{\eta})\right)^2.$$

Thus

$$\begin{aligned}\text{IV} &\le \frac{CL}{k_0^2}\frac{\left|\overline{\Omega}_{k-N}\right|^2}{\left|\Omega_k\right|^2}\sum_{\overline{\eta}}\overline{\mu}(\overline{\eta})\sum_{y,z\in\Lambda\setminus\mathcal{C}}\left(\nabla_{y,z}\overline{f}(\overline{\eta})\right)^2\\ &= \frac{CL}{k_0^2}\frac{\left|\overline{\Omega}_{k-N}\right|}{\left|\Omega_k\right|}\sum_{\eta}\mu(\eta)\mathbb{1}_{\mathcal{C}\text{ is empty}}\sum_{y,z\in\Lambda\setminus\mathcal{C}}\left(\nabla_{y,z}f(\eta)\right)^2\\ &\le \frac{CL}{k_0^2}\sum_{\eta}\mu(\eta)\mathbb{1}_{\mathcal{C}\text{ is empty}}\sum_{y,z\in\Lambda\setminus\mathcal{C}}\left(\nabla_{y,z}f(\eta)\right)^2.\end{aligned}$$

In order to conclude we need to construct a multistep move that exchanges x and y:

Claim 5.14. Fix $y,z\in\Lambda\setminus\mathcal{C}$. Then there exists a T-step move $M_{y,z}=((\eta_t),(x_t),(e_t))$ such that:

(1) $\mathrm{Dom}\,M_{y,z}=\{\eta\in\Omega_k : \mathcal{C}\text{ is empty}\}$.
(2) $M_{y,z}$ is compatible with the transposition of x and y.
(3) $T\le CL$.
(4) $\mathrm{Loss}\,M_{y,z}=0$.
(5) Each site $x\in\Lambda$ is exchanged at most $C\lambda^C$ times. Moreover,

$$|\{t\text{ such that }x_t(\eta)=x\text{ for some }\eta\in\mathrm{Dom}\,M_{s,z}\}|\le C\lambda^C.$$

Proof. If y and z are both larger than λ, the construction follows the exact same steps as that of $\overline{M}$ in the proof of Lemma 4.2.

If $y\in[\lambda]$, we perform the following maneuver—first, move the cluster 3λ steps to the right. This move is compatible with some permutation σ. Since the order of the particles is conserved in one dimension, $\sigma(y)$ and $\sigma(y+\lambda)$ are both in $[3\lambda]$. We can then exchange them using the cluster at $3\lambda+\mathcal{C}$ by the same construction as Lemma 4.2. When we now move the cluster back to the left, the net result is a move compatible with transposing y and $y+\lambda$.

If $z>\lambda$ we can apply the move constructed in the beginning, exchaning $y+\lambda$ with z, and finally wind back our manoeuvre to exchange y and $y+\lambda$. This leaves us with the configuration $\eta^{y,z}$ as we wanted.

If z is also in $[l]$, we move the cluster 2λ steps to the right. Then use it to exchange $\sigma(y)$ and $\sigma(z)$. Then move the cluster back 2λ steps to the left.

If L is large enough all these maneuvers take negligible time, and we are left with the bound $T \leq CL$. □

We can now use this newly constructed move $M_{y,z} = ((\eta_t^{y,z}), (x_t^{y,z}), (e_t^{y,z}))$ in order to finish the bound on IV:

$$\begin{aligned}\text{IV} &\leq \frac{CL^2}{k_0^2} \sum_{\eta} \mu(\eta) \mathbb{1}_{\mathcal{C} \text{ is empty}} \sum_{y,z \in \Lambda \setminus \mathcal{C}} \sum_{t=0}^{T-1} \sum_{\eta'} \sum_{x \in \Lambda} \mathbb{1}_{\eta' = \eta_t^{y,z}} \mathbb{1}_{x = x_t^{y,z}} c_{x,x+1}(\eta') \left(\nabla_{x,x+1} f(\eta')\right)^2 \\ &= \frac{CL^4 \lambda^C}{k_0^2} \sum_{\eta'} \mu(\eta') \sum_{x \in \Lambda} c_{x,x+1}(\eta') \left(\nabla_{x,x+1} f(\eta')\right)^2 = \frac{CL^4 \lambda^C}{k_0^2} \mathcal{D} f.\end{aligned}$$

To sum it all up, assuming L is large enough and using the fact that $k_0 \geq q^C L$,

$$\text{II} \leq \text{III} + \text{IV} \leq C q^{-C} L \, \mathcal{D} f + \frac{CL^4 \lambda^C}{k_0^2} \mathcal{D} f \leq C q^{-C} \, L^2 \, \mathcal{D} f.$$

We have thus proven inequalities (5.2) and (5.3), concluding the proof of Theorem 5.6. □

6. Diffusion coefficient

In this section we consider the model on $\mathbb{Z}^d$, and study the diffusion coefficient D. This is a symmetric matrix given by the following variational formula (see, e.g., [27, II.2.2]): for any $u \in \mathbb{R}^d$,

$$u \cdot Du = \frac{1}{2q(1-q)} \inf_f \mu \left[\sum_{\alpha=1}^{d} c_{0,e_\alpha} \left(u \cdot e_\alpha (\eta(0) - \eta(e_\alpha)) + \sum_x \nabla_{0,e_\alpha} \tau_x f \right)^2 \right]. \tag{6.1}$$

In [11], convergence to a hydrodynamic limit of a variation of Example 2.1 is proven, and the diffusion coefficient is found explicitly. This is done by a careful choice of the rates, rendering the model *gradient*. Proving convergence to a hydrodynamic limit for Example 2.1 with the original rates, and identifying the diffusion coefficient, is a much more difficult task. However, equation (6.1), together with the result of [11], allows us to deduce the positivity of the diffusion coefficient, and even give an estimate accurate up to a factor (to be precise, $q \leq D \leq 2q$).

In this section we prove the positivity of the diffusion coefficient in a much more general setting, for all noncooperative models.

Theorem 6.1. *Consider a noncooperative kinetically constrained lattice gas, and let D be the associated diffusion coefficient (given in equation (6.1)). Then D is positive definite, that is, $u \cdot Du$ is strictly positive for any $u \in \mathbb{R}^d$.*

Remark 6.2. The proof of Theorem (6.1) also provides bounds on the diffusion coefficient, and in particular shows that it could decay at most polynomially fast as q tends to 0. This power law behavior is characteristic of noncooperative models, while cooperative models are expected to show faster decay (see e.g. [25]).

6.1. **Proof.**

6.1.1. *Comparison argument.* We will see here how to bound the diffusion coefficient using multistep moves that compare our model to an auxiliary dynamics. For this purpose, consider the dynamics defined by a generator

$$\mathcal{L}_{\text{aux}} f = \sum_{x \sim y} c^{\text{aux}}_{x,y}(\eta) \nabla_{x,y} f(\eta), \tag{6.2}$$

and assume:

(1) The rates $c^{\text{aux}}_{x,y}(\eta)$ do not depend on $\eta(x)$, $\eta(y)$. This guarantees that the dynamics is reversible with respect to μ.
(2) The model is translation invariant.
(3) The rates are bounded from above by $c^{\text{aux}}_{\text{max}}$.

In order to compare the two models, we need to be able to perform the exchanges of the auxiliary model using the original dynamics. This will be done using a multistep move:

Hypothesis 6.3. For any $\alpha \in \{1, \dots, d\}$ there exists a T_{Aux}-step move Aux_α such that:

(1) $\text{Dom}(\text{Aux}_\alpha) = \left\{\eta \in \Omega : c^{\text{aux}}_{0,e_\alpha}(\eta) \neq 0\right\}$,
(2) The move is compatible with the permutation exchanging 0 and e_α.
(3) $x_t \in \Lambda$ for all t, where Λ is a fixed set.

Lemma 6.4. *Consider the auxiliary model (6.2), and let D^{aux} be its diffusion coefficient. If Hypothesis (6.3) is satisfied, then for any $u \in \mathbb{R}^d$*

$$u \cdot D^{aux} u \leq d T^2_{Aux} 2^{\text{Loss}(Aux)} c^{aux}_{max} |\Lambda| \; u \cdot Du.$$

Proof. Fix a local function $f : \Omega \to \mathbb{R}$. We need to show that

$$\sum_{\alpha=1}^{d} \mu \left[c^{\text{aux}}_{0,e_\alpha} \left(u \cdot e_\alpha (\eta(0) - \eta(e_\alpha)) + \sum_x \nabla_{0,e_\alpha} \tau_x f \right)^2 \right]$$
$$\leq d T^2_{\text{Aux}} 2^{\text{Loss(Aux)}} c^{\text{aux}}_{\text{max}} |\Lambda| \sum_{\alpha=1}^{d} \mu \left[c_{0,e_\alpha} \left(u \cdot e_\alpha (\eta(0) - \eta(e_\alpha)) + \sum_x \nabla_{0,e_\alpha} \tau_x f \right)^2 \right].$$

Fix α, and denote $\text{Aux}_\alpha = ((\eta_t), (x_t), (e_t))$. Then, for $\eta \in \text{Dom}(\text{Aux}_\alpha)$ we can write

$$u \cdot e_\alpha \left(\eta(0) - \eta(e_\alpha)\right) = \sum_{t=0}^{T-1} u \cdot e_t \; \left(\eta_t(x_t) - \eta_t(x_t + e_t)\right),$$
$$\nabla_{0,e_\alpha} \tau_x f = \sum_{t=0}^{T-1} \nabla_{x_t, x_t + e_t} \, \tau_x f(\eta_t).$$

Using these equalities,

$$\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(u\cdot e_\alpha(\eta(0)-\eta(e_\alpha))+\sum_x \nabla_{0,e_\alpha}\tau_x f\right)^2\right]$$

$$=\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\sum_{t=0}^{T} u\cdot e_t\left(\eta_t(x_t)-\eta_t(x_t+e_t)\right)+\sum_x\sum_{t=0}^{T}\nabla_{x_t,x_t+e_t}\tau_x f\right)^2\right]$$

$$\le T_{\text{Aux}}\mu\left[c_{0,e_\alpha}^{\text{aux}}\sum_{t=0}^{T}\left(u\cdot e_t\left(\eta_t(x_t)-\eta_t(x_t+e_t)\right)+\sum_x \tau_{x_t}\nabla_{0,e_t}\tau_{-x_t}\tau_x f\right)^2\right]$$

$$=T_{\text{Aux}}\mu\left[c_{0,e_\alpha}^{\text{aux}}\sum_{t=0}^{T}c_{x_t,x_t+e_t}(\eta_t)\left(u\cdot e_t\tau_{x_t}\left(\eta_t(0)-\eta_t(e_t)\right)+\tau_{x_t}\sum_x\nabla_{0,e_t}\tau_x f\right)^2\right]$$

$$=T_{\text{Aux}}\sum_\eta \mu(\eta)c_{0,e_\alpha}^{\text{aux}}\sum_{t=0}^{T}\sum_{z\in\Lambda}\mathbb{1}_{z=x_t}\sum_{\eta'}\mathbb{1}_{\eta'=\tau_z\eta_t}\sum_{\alpha'}\mathbb{1}_{e_{\alpha'}=e_t}c_{0,e_{\alpha'}}(\eta')$$

$$\times\left(u\cdot e_{\alpha'}\left(\eta'(0)-\eta'(e_{\alpha'})\right)+\sum_x\nabla_{0,e_{\alpha'}}\tau_x f(\eta')\right)^2$$

$$=T_{\text{Aux}}^2 2^{\text{Loss(Aux)}}c_{\max}^{\text{aux}}\left|\Lambda\right|\sum_{\eta'}\mu(\eta')c_{0,e_{\alpha'}}(\eta')\sum_{\alpha'}\left(u\cdot e_{\alpha'}\left(\eta'(0)-\eta'(e_{\alpha'})\right)+\sum_x\nabla_{0,e_{\alpha'}}\tau_x f(\eta')\right)^2$$

$$=T_{\text{Aux}}^2 2^{\text{Loss(Aux)}}c_{\max}^{\text{aux}}\left|\Lambda\right|\sum_{\alpha'=1}^{d}\mu\left[c_{0,e_{\alpha'}}\left(u\cdot e_{\alpha'}\left(\eta(0)-\eta(e_{\alpha'})\right)+\sum_x\nabla_{0,e_{\alpha'}}\tau_x f\right)^2\right].$$

□

6.1.2. *The auxiliary model.* We now define an auxiliary model that will satisfy Hypothesis 6.3. In order to do that, fix d finite sets of sites, $\mathcal{A}^\alpha=\left\{x_1^\alpha,\dots,x_{n_\alpha}^\alpha\right\}$ for $\alpha\in\{1,\dots,d\}$. We order $x_1^\alpha,\dots,x_{n_\alpha}^\alpha$ from right to left according to their α coordinate, so that $x_i^\alpha\cdot e_\alpha\ge x_j^\alpha\cdot e_\alpha$ if $i\le j$. We also define the sets

$$\mathcal{A}_i^\alpha=\left\{x_j^\alpha+e_\alpha\,,\,1\le j\le i\right\}\cup\left\{x_j^\alpha\,,\,i+1\le j\le n_\alpha\right\}$$

for $i\in\{0,\dots,n_\alpha\}$, so that $\mathcal{A}_0^\alpha=\mathcal{A}^\alpha$, and $\mathcal{A}_{i+1}^\alpha$ is obtained from $\mathcal{A}_i^\alpha$ by moving x_{i+1}^α one step in the direction e_α. Note that thanks to the ordering we have chosen, the new site $x_i^\alpha+e_\alpha$ does not belong to $\mathcal{A}_i^\alpha$, so that $|\mathcal{A}_i^\alpha|=n_\alpha$ for all i, and $\mathcal{A}_{n_\alpha}^\alpha=\mathcal{A}^\alpha+e_\alpha$.

We will now define a Markov process on Ω with the aid of these sets. The idea would be to allow empty copies of $\mathcal{A}^\alpha$ to move in the direction $\pm e_\alpha$, vacancy by vacancy, by changing at each step $\mathcal{A}_i^\alpha$ to $\mathcal{A}_{i\pm1}^\alpha$. More precisely, for each α and each $i\in\{0,\dots,n_\alpha-1\}$, we identify all translations of $\mathcal{A}_i^\alpha$ of the form $x+\mathcal{A}_i^\alpha$ which are empty for η. Then, with rate 1, we exchange sites $x+x_{i+1}^\alpha$ and $x+x_{i+1}^\alpha+e_\alpha$. In addition, for each α and each $i\in\{1,\dots,n_\alpha\}$, we identify all translations of $\mathcal{A}_i^\alpha$ of the form $x+\mathcal{A}_i^\alpha$ which are empty for η. Then, with rate 1, we exchange sites $x+x_i^\alpha$ and $x+x_i^\alpha+e_\alpha$. This could be described using the following infinitesimal generator

operating on a local function f:

$$\begin{aligned}\mathcal{L}_{\text{aux}} f &= \sum_{\alpha=1}^{d}\sum_{i=0}^{n_\alpha-1}\sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} \nabla_{x+x_{i+1}^\alpha, x+x_{i+1}^\alpha+e_\alpha} f(\eta) \\ &\quad + \sum_{\alpha=1}^{d}\sum_{i=1}^{n_\alpha}\sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} \nabla_{x+x_i^\alpha, x+x_i^\alpha+e_\alpha} f(\eta).\end{aligned} \tag{6.3}$$

We will refer to the transition described in the first sum as *forward transitions*, and to the ones in the second sum as *backward transitions*. That is, a forward transition occurs when an empty site x is exchanged with an occupied neighbor $x+e_\alpha$, and a backward transition occurs when an empty site y is exchanged with an occupied neighbor $y-e_\alpha$. Note that a forward transition from x to $x+e_\alpha$ is only possible when for some $\tilde{x}\in\mathbb{Z}^2$ and $i\in\{0,\dots,n_\alpha-1\}$, $\tilde{x}+\mathcal{A}_i^\alpha$ is empty and $x=\tilde{x}+x_{i+1}^\alpha$. In other words, we need $x-x_{i+1}^\alpha+\mathcal{A}_i^\alpha$ to be empty for some $i\in\{0,\dots,n_\alpha-1\}$. Similarly, a backward transition from y to $y-e_\alpha$ requires $y-e_\alpha-x_i^\alpha+\mathcal{A}_i^\alpha$ to be empty for some $i\in\{1,\dots,n_\alpha\}$.

Observation 6.5. The auxiliary dynamics (6.3) is reversible with respect to the equilibrium measure μ, for any value of the parameter q.

Proof. This is a consequence of the fact that for any $\eta\in\Omega$ and any edge $x\sim y$ of $\mathbb{Z}^2$, the rate at which η changes to $\eta^{x,y}$ is the same as the rate at which $\eta^{x,y}$ changes to η—without loss of generality assume $\eta(x)=1-\eta(y)=0$ and $y=x+e_\alpha$. Then the rate of exchanging x and y for η is given by the number of sets $\mathcal{A}_i^\alpha$, $i\in\{0,\dots,n_\alpha-1\}$, such that $x-x_{i+1}+\mathcal{A}_i^\alpha$ is empty for η. On the other hand, the rate of exchanging x and y for $\eta^{x,y}$ is given by the number of sets $\mathcal{A}_i^\alpha$, $i\in\{1,\dots,n_\alpha\}$, such that $y-e_\alpha-x_i+\mathcal{A}_i^\alpha$ is empty for $\eta^{x,y}$. The latter could be written as

$$\begin{aligned}&\#\left\{i\in\{1,\dots,n_\alpha\} : y-e_\alpha-x_i+\mathcal{A}_i^\alpha \text{ is empty for } \eta^{x,y}\right\} \\ &= \#\left\{i\in\{0,\dots,n_\alpha-1\} : x-x_{i+1}+\mathcal{A}_{i+1}^\alpha \text{ is empty for } \eta^{x,y}\right\} \\ &= \#\left\{i\in\{0,\dots,n_\alpha-1\} : x-x_{i+1}+\mathcal{A}_i^\alpha \text{ is empty for } \eta\right\},\end{aligned}$$

which conclude the proof. □

The last observation shows that $\mathcal{L}_{\text{aux}}$ could be put in the form (6.2), where the rates $c_{x,y}^{\text{aux}}$ are bounded by n_α.

The key property of this model is that the total current vanishes for any configuration:

Observation 6.6. Consider the auxiliary dynamics (6.3) on the torus $\mathbb{Z}^d/L\mathbb{Z}^d$, for some fixed (large) L. Then, for any $\eta\in\Omega$, the total current is 0. That is,

$$\sum_{x\sim y} c_{x,y}^{\text{aux}}\,(x-y)\,(\eta(x)-\eta(y))=0.$$

Proof. Fix $\alpha \in \{1, \dots, d\}$. We show that the total current in the α direction is 0. The negative current (particles moving in the direction $-e_\alpha$) is given by forward transitions, and the positive current by backward transitions. We need to prove that the two cancel out.

Each empty translation of $\mathcal{A}_i^\alpha$ contributes a forward transition of rate 1, unless we try to move the vacancy to an already empty site. Hence the rate of forward transitions is given by

$$\begin{aligned}
&\sum_{i=0}^{n_\alpha-1} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} - \sum_{i=0}^{n_\alpha-1} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} \mathbb{1}_{x+x_{i+1}^\alpha+e_\alpha \text{ is empty}} \\
&\quad = \sum_{i=1}^{n_\alpha-1} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} + \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_0^\alpha \text{ are empty}} - \sum_{i=0}^{n_\alpha-1} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_{i+1}^\alpha \text{ are empty}} \mathbb{1}_{x+x_{i+1}^\alpha \text{ is empty}} \\
&\quad = \sum_{i=1}^{n_\alpha-1} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} + \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_{n_\alpha}^\alpha \text{ are empty}} - \sum_{i=1}^{n_\alpha} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} \mathbb{1}_{x+x_i^\alpha \text{ is empty}} \\
&\quad = \sum_{i=1}^{n_\alpha} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} - \sum_{i=1}^{n_\alpha} \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{x+\mathcal{A}_i^\alpha \text{ are empty}} \mathbb{1}_{x+x_i^\alpha \text{ is empty}}.
\end{aligned}$$

We recognize the last line as the rate of backward transitions, which finishes the proof. □

The zero current property, as explained in [27, II.2.4], makes the contribution of the current-current correlation to the diffusion coefficient vanish. This allows us to calculate explicitly the diffusion coefficient.

Lemma 6.7. *Let D^{aux} be the diffusion coefficient associated to the auxiliary dynamics (6.3). Then for any $u \in \mathbb{R}^d$*

$$u \cdot D^{aux} u = \sum_{\alpha=1}^{d} (u \cdot e_\alpha)^2 \, \mu\left[c_{0,e_\alpha}\right] \geq C q^n \, \|u\|^2,$$

where $n = \max_\alpha n_\alpha$.

Proof. The inequality follows directly from the definition of the model, so we are left with showing the equality. [27, II.2.4] explains how it could be derived from the Green-Kubo formula [27, II, equation (2.27)], for completeness we will prove it explicitly from the variational characterization (6.1).

Fix a local function f, and L large enough (depending on the support of f), so that $\sum_{x\in\mathbb{Z}^d}$ in equation (6.1) could be replaced by $\sum_{x\in\mathbb{Z}^d/L\mathbb{Z}^d}$. Then

$$\begin{aligned}
&\mu\left[\sum_{\alpha=1}^{d} c_{0,e_\alpha}^{\text{aux}} \left(u \cdot e_\alpha(\eta(0) - \eta(e_\alpha)) + \sum_x \nabla_{0,e_\alpha} \tau_x f\right)^2\right] \\
&\quad = \sum_{\alpha=1}^{d} \mu\left[c_{0,e_\alpha}^{\text{aux}} \left(u \cdot e_\alpha(\eta(0) - \eta(e_\alpha))\right)^2\right] + 2 \sum_{\alpha=1}^{d} \mu\left[c_{0,e_\alpha}^{\text{aux}} \, u \cdot e_\alpha(\eta(0) - \eta(e_\alpha)) \sum_x \nabla_{0,e_\alpha} \tau_x f\right]
\end{aligned}$$

$$+\sum_{\alpha=1}^{d}\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\sum_x \nabla_{0,e_\alpha}\tau_x f\right)^2\right]$$

$$\geq \sum_{\alpha=1}^{d}\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(u\cdot e_\alpha(\eta(0)-\eta(e_\alpha))\right)^2\right]+2\sum_{\alpha=1}^{d}u\cdot e_\alpha\sum_x\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\eta(0)-\eta(e_\alpha)\right)\nabla_{0,e_\alpha}\tau_x f\right].$$

Since μ is invariant under the map $\eta\mapsto\eta^{0,e_\alpha}$ and $c_{0,e_\alpha}^{\text{aux}}(\eta)=c_{0,e_\alpha}^{\text{aux}}(\eta^{0,e_\alpha})$, we can write for any function g

$$\begin{aligned}\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\eta(0)-\eta(e_\alpha)\right)g(\eta)\right]&=\frac{1}{2}\left(\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\eta(0)-\eta(e_\alpha)\right)g(\eta)\right]+\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\eta^{0,e_\alpha}(0)-\eta^{0,e_\alpha}(e_\alpha)\right)g(\eta^{0,e_\alpha})\right]\right)\\&=-\frac{1}{2}\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\eta(0)-\eta(e_\alpha)\right)\nabla_{0,e_\alpha}g(\eta)\right].\end{aligned}$$

Therefore, setting $g=\tau_x f$ and then using the translation invariance of μ we obtain

$$\begin{aligned}\sum_x\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\eta(0)-\eta(e_\alpha)\right)\nabla_{0,e_\alpha}\tau_x f\right]&=-2\sum_x\mu\left[c_{0,e_\alpha}^{\text{aux}}\left(\eta(0)-\eta(e_\alpha)\right)\tau_x f\right]\\&=-2\sum_x\mu\left[c_{x,x+e_\alpha}^{\text{aux}}\left(\eta(x)-\eta(x+e_\alpha)\right)f\right]=-2\mu\left[\left(\sum_x c_{x,x+e_\alpha}^{\text{aux}}\left(\eta(x)-\eta(x+e_\alpha)\right)\right)f\right].\end{aligned}$$

The last term is 0 by Observation (6.6), proving that

$$u\cdot D^{\text{aux}}u\geq\frac{1}{2q(1-q)}\,\mu\left[\sum_{\alpha=1}^{d}c_{0,e_\alpha}^{\text{aux}}\left(u\cdot e_\alpha(\eta(0)-\eta(e_\alpha))\right)^2\right].$$

Hence, the infimum in equation (6.1) is attained for constant f.

Finally, we use the product structure of μ and the fact that $c_{0,e_\alpha}^{\text{aux}}$ does not depend on $\eta(0)$ and $\eta(e_\alpha)$ to calculate this infimum explicitly:

$$\begin{aligned}u\cdot D^{\text{aux}}u&=\frac{1}{2q(1-q)}\,\mu\left[\sum_{\alpha=1}^{d}c_{0,e_\alpha}\left(u\cdot e_\alpha(\eta(0)-\eta(e_\alpha))\right)^2\right]\\&=\frac{1}{2q(1-q)}\sum_{\alpha=1}^{d}(u\cdot e_\alpha)^2\mu\left[c_{0,e_\alpha}\right]\left[\left(\eta(0)-\eta(e_\alpha)\right)^2\right]\\&=\sum_{\alpha=1}^{d}(u\cdot e_\alpha)^2\mu\left[c_{0,e_\alpha}\right].\end{aligned}$$

□

6.1.3. *The multistep move.* As a corollary of lemmas 6.4 and 6.7, if we assume that for any α there exists $\mathcal{A}^\alpha$ of size $n_\alpha\leq n$ such that the auxiliary model defined in (6.3) satisfies Hypothesis (6.3), then

$$u\cdot Du\geq Cq^n\,\|u\|^2$$

for any $u\in\mathbb{R}^d$.

Example 6.8. In Example 2.1, we may take $\mathcal{A}_0 = \{1, 2\}$ so $\mathcal{A}_1 = \{1, 3\}$ and $\mathcal{A}_2 = \{2, 3\}$. Then the multistep Aux could be chosen trivially as the 1-step move exchanging the corresponding sites.

Similarly, in Example 2.1, we take $\mathcal{A}^1 = \{e_1, 2e_1\}$ and $\mathcal{A}^2 = \{e_2, 2e_2\}$, and verify that we may choose the trivial 1-step moves.

In these two examples we know that by modifying the rates (without changing the constrained and unconstrained transitions) as in [11] we obtain a gradient model (which is, in fact, the auxiliary model we defined above). That is, equation (6.1) could be used directly, without passing through the comparison argument. This is expressed in the fact that our multistep move is in fact a 1-step move.

In order to prove Theorem 6.1 all that is left is to construct $\mathcal{A}^\alpha$ and the Aux_α move. Consider a mobile cluster $\mathcal{C}$, and l such that $\mathcal{C} \in [l-1]^d$. Choosing, for any α, the set $\mathcal{A}^\alpha = \mathcal{C} \cup (le_\alpha + \mathcal{C})$ (with $n_\alpha = 2\,|\mathcal{C}|$) will suffice. In order to show that, we need to construct the Aux_α move.

Let $\eta \in \mathrm{Dom}\,\mathrm{Aux}_\alpha$, i.e., $c^{\mathrm{aux}}_{0,e_\alpha} > 0$. By reversibility we may assume that this is a forward transition, so $\eta(0) = 1 - \eta(e_\alpha) = 0$, and there exists $i \in \{0, \dots, n_\alpha - 1\}$ such that $-x_{i+1} + \mathcal{A}^\alpha_i$ is empty. We consider two cases:

Case 1. $i \in \{0, \dots, |\mathcal{C}| - 1\}$. Then $-x_{i+1} + \mathcal{C} = -x_{i+1} + \{x_{|\mathcal{C}|+1}, \dots, x_{n_\alpha}\} \subset \mathcal{A}^\alpha_i$. Moreover, neither 0 nor e_α are contained in $-x_{i+1} + \mathcal{C}$ since $x_{i+1} \in le_\alpha + [l-1]^d$. We may therefore apply translation and exchange moves using the mobile cluster $-x_{i+1} + e_\alpha + le_\alpha + \mathcal{C}$ in order to exchange 0 and e_α.

Case 2. $i \in \{|\mathcal{C}|, \dots, n_\alpha\}$. Then $-x_{i+1} + e_\alpha + le_\alpha + \mathcal{C} = -x_{i+1} + e_\alpha + \{x_1, \dots, x_{|\mathcal{C}|}\} \subset \mathcal{A}^\alpha_i$. As before, neither 0 nor e_α are contained in $-x_{i+1} + e_\alpha + le_\alpha + \mathcal{C}$ since $x_{i+1} \in [l-1]^d$. We may therefore apply translation and exchange moves using the mobile cluster $-x_{i+1} + e_\alpha + le_\alpha + \mathcal{C}$ in order to exchange 0 and e_α.

Hypothesis 6.3 is thus satisfied, concluding the proof of Theorem 6.1 by lemmas 6.4 and 6.7. □

Remark 6.9. While the construction above gives a polynomial bound for all noncooperative models, in specific cases it might not be optimal. In Example 2.2, the mobile cluster has size 4, therefore the estimate we obtain is of the order q^8. We have seen, however, that there is a more efficient explicit choice of $\mathcal{A}^\alpha$ which yields a much better bound, of the order q^2.

7. Self-diffusion in $d \geq 2$

In this section we study the self-diffusion coefficient D_s, which is a symmetric matrix given by the following variational formula ([26], [27, II.6.2]): for any $u \in \mathbb{R}^d$,

$$u \cdot D_s u = \frac{1}{2} \inf_f \mu_0 \left[\sum_{\substack{y \sim x \\ x,y \neq 0}} c_{xy} (\nabla_{xy} f)^2 + \sum_{y \sim 0} c_{0y}(1 - \eta(y)) \left(u \cdot y + f(\tau_{-y}\eta^{0y}) - f(\eta) \right)^2 \right]. \quad (7.1)$$

In dimension 1, due to the preservation of the order of particles, the self-diffusion coefficient is 0 even with in an unconstrained setting (see, e.g., [27, II.6.4]), we will therefore consider here only the higher dimensional case.

The positivity of the diffusion coefficient for examples 2.1 and 2.2 was proven in [2]. We will see here that it is positive for any noncooperative models.

Theorem 7.1. *Consider a noncooperative kinetically constrained lattice gas in dimension 2 or higher, and let D_s be the associated self-diffusion coefficient (given in equation (7.1)). Then D_s is positive definite, that is, $u \cdot D_s u$ is strictly positive for any $u \in \mathbb{R}^d$.*

Remark 7.2. As for the diffusion coefficient, the proof of Theorem 7.1 also shows that the rate at which D_s decays to 0 when q approaches 0 is at most polynomial, as expected.

7.1. **Proof.** The proof will follow the strategy of [27, II.6.3], also used in [2]. It consists of comparing the model to as auxiliary model where the tracer motion could be more easily tracked. The auxiliary model we will choose, however, does not fall under the framework of equation (7.1)—First, the transitions are not single particle jumps, but a simultaneous rearrangement of several particles. Moreover, these transitions are not homogeneous; more precisely, the allowed transitions and their rates depend on the position as seen from the tracer.

We start by generalizing equation (7.1) in a setting which will cover our auxiliary model. Consider a dynamics on the space of configuration Ω with additional information on the location of the tracer $z \in \mathbb{Z}^d$. Fix a countable set Σ of permutations of the sites, and assume that they all have finite range. This means that, for some fixed R, any permutation $\sigma \in \Sigma$ fixes the sites outside $x + [-R, R]^d$, where $x \in \mathbb{Z}^d$ may depend on σ. Then, for each $\sigma \in \Sigma$, we apply σ with rate $\hat{c}_\sigma$, relative to the tracer position z. That is, the configuration η becomes $\tau_z \sigma \tau_{-z} \eta$ and the tracer moves to $\tau_z \sigma \tau_{-z}(z) = z + \sigma(0)$, with rate $\hat{c}_\sigma(\tau_{-z}\eta)$. It is important to note that in the new configuration, if the old tracer position is occupied then so is the new one. This process can be written using the infinitesimal generator operating on $f : \mathbb{Z}^d \times \Omega \to \mathbb{R}$:

$$\hat{\mathcal{L}} f(z, \eta) = \sum_{\sigma \in \Sigma} \hat{c}_\sigma(\tau_{-z}\eta) \left(f(z + \sigma(0), \tau_z \sigma \tau_{-z} \eta) - f(z, \eta) \right), \quad (7.2)$$

for a set of rates $\hat{c}_\sigma : \Omega \to [0, \infty)$ defined for all any $\sigma \in \Sigma$.

Remark 7.3. To obtain the original kinetically constrained model we take Σ to be the set of nearest neighbor transpositions Σ_{kc}, and the rate

$$\hat{c}^{\text{kc}}_{(x,y)}(\eta) = \begin{cases} c_{x,y}(\tau_{-z}\eta)\mathbb{1}_{\eta(y)=0} & \text{if } x=0, \\ c_{x,y}(\tau_{-z}\eta)\mathbb{1}_{\eta(x)=0} & \text{if } y=0, \\ c_{x,y}(\tau_{-z}\eta) & \text{otherwise.} \end{cases}$$

The reason that we do not simply take $\hat{c}^{\text{kc}}_{(x,y)}(\eta) = c_{x,y}(\tau_{-z}\eta)$ is that, while in the original dynamics exchanging two particles is equivalent to doing nothing, when following the tracer we are not allowed to exchange it with a particle.

Then

$$\begin{aligned} \hat{\mathcal{L}}^{\text{kc}} f(z,\eta) &= \sum_{\substack{x\sim y \\ x,y\neq 0}} c_{x,y}(\tau_{-z}\eta)\left(f(z,\eta^{x+z,y+z}) - f(z,\eta)\right) + \sum_{0\sim y} c_{0,y}(\tau_{-z}\eta)\left(f(y,\eta^{z,y+z}) - f(z,\eta)\right) \\ &= \sum_{\substack{x\sim y \\ x,y\neq z}} c_{x,y}(\eta)\left(f(z,\eta^{x,y}) - f(z,\eta)\right) + \sum_{z\sim y} c_{z,y}(\eta)\left(f(y,\eta^{z,y}) - f(z,\eta)\right), \end{aligned}$$

which is indeed the generator of the dynamics (2.1) together with a tracer.

The variational formula (7.1) could be generalized to the setting of (7.2):

Lemma 7.4. *Consider the dynamics (7.2). Assume that, ignoring the tracer, it is reversible with respect to a probability measure ν on Ω (i.e., $\hat{\mathcal{L}}$ is self adjoint operating on functions that do not depend on z). Let ν_0 be the measure ν, conditioned on having a particle at the origin, i.e., $\nu_0(\zeta\in\cdot) = \nu(\zeta\in\cdot|\zeta(0)=1)$. Then for any $u\in\mathbb{R}^d$,*

$$u\cdot\hat{D}_s u = \frac{1}{2}\inf_f \left\{ \sum_{\sigma\in\Sigma} \nu_0\left[\hat{c}_\sigma(\zeta)\left(u\cdot\sigma(0) + f(\tau_{-\sigma(0)}\sigma\zeta) - f(\zeta)\right)^2\right]\right\},$$

where $\hat{D}_s$ is the associated self-diffusion coefficient and the infimum is taken over all local functions on $\Omega_0 = \{\zeta\in\Omega : \zeta(0)=1\}$.

Remark 7.5. From the last lemma we can reconstruct equation (7.1): as in Remark 7.3,

$$\begin{aligned} &\sum_{\sigma\in\Sigma} \nu_0\left[\hat{c}^{\text{kc}}_\sigma(\zeta)\left(f(\tau_{-\sigma(0)}\sigma\zeta) - f(\zeta) - u\cdot\sigma(0)\right)^2\right] \\ &= \sum_{\substack{x\sim y \\ x,y\neq 0}} \nu_0\left[c_{x,y}(\zeta)\left(f(\zeta^{x,y}) - f(\zeta)\right)^2\right] + \sum_{y\sim 0}\nu_0\left[c_{0,y}(\zeta)(1-\eta(y))\left(u\cdot y + f(\tau_{-y}\zeta^{0,y}) - f(\zeta)\right)^2\right]. \end{aligned}$$

Proof. The proof follows the exact same argument as [26, 27]. For completeness we present here the main steps.

Consider the process described above, with η_t and z_t the configuration and tracer position at time t. Define $\zeta_t = \tau_{-z}\eta_t$, so the joint process (ζ_t, z_t) is Markovian with generator operating

on $f : \Omega_0 \times \mathbb{Z}^d \to \mathbb{R}$ as

$$\overline{\mathcal{L}} f(\zeta, z) = \sum_{\sigma \in \Sigma} \hat{c}_\sigma(\zeta) \left(f(z + \sigma(0), \tau_{-\sigma(0)} \sigma \zeta) - f(z, \zeta) \right).$$

Fix $g(z, \zeta) = u \cdot z$, and let

$$j_u(\zeta) = \overline{\mathcal{L}} g(z, \zeta) = \sum_{\sigma \in \Sigma} \hat{c}_\sigma(\zeta)\, u \cdot \sigma(0).$$

Then

$$u \cdot z_t - \int_0^t j_u(\zeta_s) \,\mathrm{d}\, s = M_t$$

is a martingale with stationary increments and quadratic variation

$$\mathbb{E}\left(M_t^2\right) = t \sum_{\sigma \in \Sigma} (u \cdot \sigma(0))^2 \, \nu_0 \left(\hat{c}_\sigma(\zeta)\right).$$

Here, and in the rest of the proof, $\mathbb{E}(\cdot)$ refers to expectation related to the process, starting from a configuration η drawn according to ν_0 and a tracer at the origin.

We obtain

$$\mathbb{E}\left[(u \cdot z_t)^2\right] = t \sum_{\sigma \in \Sigma} (u \cdot \sigma(0))^2 \, \nu_0 \left(\hat{c}_\sigma\right) - \int_0^t \int_0^t \mathbb{E}\left[j_u(\zeta_s) j_u(\zeta_{s'})\right] \mathrm{d}\, s \,\mathrm{d}\, s' + \mathbb{E}\left[u \cdot z_t \int_0^t j_u(\zeta_s) \,\mathrm{d}\, s\right].$$

By reversibility and translation invariance, the process $(-z_{t-s}, \zeta_{t-s})_{s \in [0,t]}$ has the same law as $(z_s, \zeta_s)_{s \in [0,t]}$ (under the initial condition $z = 0$ and ζ draws from ν_0). Therefore, the last term in the equation above vanishes, leaving us with

$$u \cdot \hat{D} s u = \frac{1}{2t} \lim_{t \to \infty} \mathbb{E}\left[(u \cdot z_t)^2\right] = \frac{1}{2} \sum_{\sigma \in \Sigma} (u \cdot \sigma(0))^2 \, \nu_0 \left(\hat{c}_\sigma\right) - \int_0^\infty \nu_0 \left[j_u e^{t\overline{\mathcal{L}}} j_u\right] \mathrm{d}\, t.$$

Note that the last expression contains only functions of the configuration ζ, without looking at the tracer position z. The process $(\zeta_t)_{t=0}^\infty$ is Markovian and reversible with respect to ν_0; therefore, with some abuse of notation, we will consider from now on $\overline{\mathcal{L}}$ as the generator of this projected process, operating on functions on Ω_0.

We may now write

$$\begin{aligned} -\int_0^\infty \nu_0 \left[j_u e^{t\overline{\mathcal{L}}} j_u \,\mathrm{d}\, t\right] &= \nu_0 \left[j_u \overline{\mathcal{L}}^{-1} j_u\right] \\ &= \inf_f \left\{-2\nu_0(j_u f) - \nu_0(f \overline{\mathcal{L}} f)\right\}. \end{aligned}$$

In order to calculate the first term in the infimum we use the detailed balance equation. For every σ, defining $\sigma' = \tau_{-\sigma(0)} \sigma^{-1} \tau_{\sigma(0)}$ (so that applying σ and then σ' brings us back to the original configuration),

$$\nu_0 \left[\hat{c}_\sigma(\zeta) f(\zeta)\right] = \nu_0 \left[\hat{c}_{\sigma'}(\zeta) f(\tau_{-\sigma'(0)} \sigma' \zeta)\right].$$

Hence, using $\sigma'(0) = -\sigma(0)$,

$$\begin{aligned} -2\nu_0\left[j_u f\right] &= -2\sum_{\sigma\in\Sigma} u\cdot\sigma(0)\,\nu_0\left[\hat{c}_\sigma(\zeta) f(\zeta)\right] \\ &= \sum_{\sigma\in\Sigma} u\cdot\sigma(0)\,\nu_0\left[\hat{c}_\sigma(\zeta)\left(f(\tau_{-\sigma(0)}\sigma\zeta) - f(\zeta)\right)\right]. \end{aligned}$$

The second term in the infimum is given by the Dirichlet form

$$-\nu_0(f\overline{\mathcal{L}}f) = \frac{1}{2}\sum_{\sigma\in\Sigma}\nu_0\left[\hat{c}_\sigma(\zeta)\left(f(\tau_{-\sigma(0)}\sigma\zeta) - f(\zeta)\right)^2\right].$$

Summing all up,

$$\begin{aligned} &\frac{1}{2}\inf_f\left\{\sum_{\sigma\in\Sigma}\nu_0\left[\hat{c}_\sigma(\zeta)\left(u\cdot\sigma(0) + f(\tau_{-\sigma(0)}\sigma\zeta) - f(\zeta)\right)^2\right]\right\} \\ &\qquad = \frac{1}{2}\sum_\sigma (u\cdot\sigma(0))^2\nu_0(\hat{c}_\sigma) + \inf_f\left\{-\nu_0(f\overline{\mathcal{L}}f) - 2\nu_0(j_u f)\right\} \\ &\qquad\qquad = \frac{1}{2}\sum_\sigma (u\cdot\sigma(0))^2\nu_0(\hat{c}_\sigma) - \int_0^\infty \nu_0\left[j_u e^{t\overline{\mathcal{L}}} j_u\,\mathrm{d}\,t\right] = u\cdot\hat{D}_s u. \quad \square \end{aligned}$$

7.1.1. *Comparison argument.* As in the case of the diffusion coefficient, we will see that an appropriate move could help us compare different dynamics.

Consider a model as in equation (7.2), satisfying the following conditions:

(1) For any $\sigma\in\Sigma$, the configuration $\sigma' = \tau_{-\sigma(0)}\sigma^{-1}\tau_{\sigma(0)}$ is also in Σ, and $\hat{c}_\sigma = \hat{c}_{\sigma'}$. This is equivalent to reversibility with respect to the equilibrium measure μ (for any q).
(2) $\hat{c}_\sigma \le 1$ for any $\sigma\in\Sigma$.

The comparison argument will be based on multistep moves, requiring us to follow the tracer position throughout the move.

Definition 7.6. Fix a T-step move $M = ((\eta_t),(x_t),(e_t))$, and assume that for any $\eta\in\mathrm{Dom}(M)$ some given site z_0 is occupied, i.e., $\eta(z_0) = 1$. Then the *tracer position associated with* M *starting at* z_0 is a sequence of sites $(z_t)_{t=0}^T$ giving at each step t the position of the particle originally at z_0:

$$z_{t+1} = \begin{cases} x_t + e_t & \text{if } z_t = x_t \text{ and } \eta_t(x_t+e_t) = 0, \\ x_t & \text{if } z_t = x_t + e_t \text{ and } \eta_t(x_t) = 0, \\ z_t & \text{otherwise.} \end{cases}$$

In order to compare the auxiliary model with our kinetically constrained lattice gas, we must have an appropriate multistep move:

Hypothesis 7.7. For any $\sigma\in\Sigma$ and $z_0\in\mathbb{Z}^d$, there is a T-step move $M_{z_0,\sigma} = ((\eta_t),(x_t),(e_t))$ such that:

(1) Dom$M = \{\eta \in \Omega : \eta(z_0) = 1 \text{ and } \hat{c}_\sigma(\tau_{-z_0}\eta) > 0\}$.
(2) M is compatible with the permutation $\tau_{z_0}\sigma\tau_{-z_0}$.
(3) In all transitions involving the tracer, the site it jumps to must be empty. More precisely, denote z_t the tracer position associated with M starting from z_0. Then, for all t, if $x_t = z_t$ then $\eta_t(x_t + e_t) = 0$ and if $x_t + e_t = z_t$ then $\eta_t(x_t) = 0$.
(4) For any z_0, t, η', x', e' and z',

$$|\{\sigma \in \Sigma : \eta_t = \eta', x_t = x', e_t = e', z_t = z'\}| \leq C.$$

We note that by translation invariance of the kinetically constrained lattice gas, it suffices to construct $M_{z_0,\sigma}$ for a specific choice of z_0 to guarantee its existence for all z_0.

Lemma 7.8. *Consider an auxiliary model as in (7.2), reversible with respect to μ and with rates bounded by 1. Assume that Hypothesis 7.7 holds. Then for all $u \in \mathbb{R}^d$,*

$$u \cdot \hat{D}_s u \leq C\, u \cdot D_s u,$$

where D_s and $\hat{D}_s$ are the self diffusion coefficients associated with the kinetically constrained lattice gas and the auxiliary model respectively.

Proof. Fix $z_0 \in \mathbb{Z}^d$ and $\sigma \in \Sigma$, and consider the move $M_{z_0,\sigma} = ((\eta_t), (x_t), (e_t))$ given in Hypothesis 7.7. Let z_t be the associated tracer position starting at z_0. Fix $\eta \in \text{Dom}\, M_{z_0,\sigma}$, and set $\zeta = \tau_{-z_0}\eta$, $\zeta_t = \tau_{-z}\eta_t$ and $\sigma_t = (x_t - z_t, x_t - z_t + e_t)$ for all t. Note first that

$$\begin{aligned} u \cdot \sigma(0) + f(\tau_{-\sigma(0)}\sigma\zeta) - f(\zeta) &= u \cdot (z_T - z_0) + f(\zeta_T) - f(\zeta_0) \\ &= \sum_{t=0}^{T-1} u \cdot (z_{t+1} - z_t) + f(\zeta_{t+1}) - f(\zeta_t). \end{aligned}$$

Also,

$$\begin{aligned} z_{t+1} &= z_t + \sigma_t(0), \\ \zeta_{t+1} &= \tau_{-\sigma_t(0)}\sigma_t\zeta_t. \end{aligned}$$

Recall remarks 7.3 and 7.5. Setting $z_0 = 0$ (and hence $\zeta = \eta$),

$$\begin{aligned} &\sum_{\sigma\in\Sigma} \mu_0 \left[\hat{c}_\sigma(\zeta) \left(\sum_{t=0}^{T-1} u \cdot (z_{t+1} - z_t) + f(\zeta_{t+1}) - f(\zeta_t) \right)^2 \right] \\ &\qquad \leq T \sum_{\sigma\in\Sigma} \mu_0 \left[\hat{c}_\sigma(\zeta) \sum_{t=0}^{T-1} \left(u \cdot \sigma_t(0) + f(\tau_{-\sigma_t(0)}\sigma_t\zeta_t) - f(\zeta_t) \right)^2 \right] \\ &\leq CT \sum_{z\in[-R,R]^d} \sum_{t=0}^{T-1} \mu_0 \left[\sum_{\sigma'\in\Sigma_{\text{kc}}} \mathbb{1}_{z'=z_t} \mathbb{1}_{\sigma'=((x_t-z',x_t-z'+e_t))} \hat{c}^{\text{kc}}_{\sigma'} \left(u \cdot \sigma'(0) + f(\tau_{-\sigma'(0)}\sigma'\zeta') - f(\zeta') \right)^2 \right] \end{aligned}$$

$$\leq CT^2R^d\mu_0\left[\sum_{\sigma'\in\Sigma_{\mathrm{kc}}}\hat{c}^{\mathrm{kc}}_{\sigma'}\left(u\cdot\sigma'(0)+f(\tau_{-\sigma'(0)}\sigma'\zeta')-f(\zeta')\right)^2\right].$$

This concludes the proof by Lemma 7.4. □

7.1.2. *The auxiliary model.* Fix some finite set $\hat{\mathcal{C}}\subset\mathbb{Z}^d\setminus\{0\}$, and d permutations $\sigma_1,\dots,\sigma_d$ with finite range. Assume that $\sigma_i(0)=e_i$ and that $\sigma_i(\hat{\mathcal{C}})=e_i+\hat{\mathcal{C}}$. For all $i\in[d]$ set

$$\sigma_{-i}=\tau_{-\sigma_i(0)}\sigma_i^{-1}\tau_{\sigma_i(0)},$$

so in particular

$$\sigma_{-i}(0)=-e_i,\quad \sigma_{-i}(\hat{\mathcal{C}})=-e_i+\hat{\mathcal{C}}.$$

We then define the auxiliary model as in equation (7.2), with $\Sigma=\{\sigma_{\pm1},\dots,\sigma_{\pm d}\}$ and $\hat{c}_\sigma(\eta)=\mathbb{1}_{\hat{\mathcal{C}}\text{ is empty}}$ for all $\sigma\in\Sigma$. It is indeed reversible with respect to μ, and all rates are bounded by 1 (as required by Lemma 6.4).

Lemma 7.9. *Consider the auxiliary model defined above. Then for all $u\in\mathbb{R}^d$*

$$u\cdot\hat{D}_su=\frac{1}{2}q^{|\hat{\mathcal{C}}|}\left\|u\right\|^2.$$

Proof. Start the dynamics with a configuration η_0 drawn from μ_0 and tracer at the origin.

Assume $\hat{\mathcal{C}}$ is empty for η_0. Then the entire cluster $\hat{\mathcal{C}}\cup\{0\}$ performs a simple random walk, independently of the initial configuration. This is because initially all rates are 1, and in each transition the tracer moves together with $\hat{\mathcal{C}}$, meaning that all rates remain 1.

On the other hand, if $\hat{\mathcal{C}}$ is not empty initially, then the configuration is blocked, and the tracer remain at the origin forever. Hence, denoting the tracer position at time t by z_t,

$$\begin{aligned}u\cdot\hat{D}_su&=\lim_{t\to\infty}\frac{1}{2t}\mathbb{E}\left((u\cdot z_t)^2\right)=\lim_{t\to\infty}\frac{1}{2t}\mathbb{E}\left((u\cdot z_t)^2\mathbb{1}_{\hat{\mathcal{C}}\text{ is empty for }\eta_0}\right)\\&=\frac{1}{2}\left\|u\right\|^2\mu(\hat{\mathcal{C}}\text{ is empty for }\eta_0).\end{aligned}$$

□

7.1.3. *The multistep move.* In this section we construct the multistep moves allowing us to move the tracer together with an empty cluster $\hat{\mathcal{C}}$.

Fix a mobile cluster $\mathcal{C}$ and $l>0$ such that the translation and exchange moves exist. We define

$$\hat{\mathcal{C}}=\{-e_1\}\cup((l+2)e_1+\mathcal{C}).$$

Claim 7.10. There exists a T-step move $\mathrm{Hop}=((\eta_t),(x_t),(e_t))$, which we call the vacancy hopping move, such that:

(1) $\mathrm{Dom}\,\mathrm{Hop}=\left\{\eta:\eta(0)=1\text{ and }\hat{\mathcal{C}}\text{ is empty}\right\}$.
(2) Hop is a deterministic move, compatible with the cyclic permutation $\sigma_{\mathrm{H}}=(e_1,e_1+e_2,e_2,-e_1+e_2,-e_1)$.
(3) For all t, at least one of the two sites x_t or x_t+e_t must be empty.

Proof. We will construct Hop as a composition of several moves. First, we use translation moves in order to bring the mobile cluster to $-e_1 - le_2 + \mathcal{C}$:

$$M_1 = \mathrm{Tr}_2(-(l+1)e_2 - e_1 + \mathcal{C}) \circ \mathrm{Tr}_{-1}(-(l+1)e_2 + \mathcal{C}) \circ \cdots \circ \mathrm{Tr}_{-1}(-(l+1)e_2 + (l+2)e_1 + \mathcal{C}) \\ \circ \mathrm{Tr}_{-2}(-le_2 + (l+2)e_1 + \mathcal{C}) \circ \cdots \circ \mathrm{Tr}_{-2}((l+2)e_1 + \mathcal{C}).$$

We emphasize that, for each of these translation $\mathrm{Tr}(x + \mathcal{C})$, the sites $-e_1, -e_1 + e_2$ are outside $x + [-l, l]$, hence untouched by the move. Also, the translation move is deterministic, and since adding vacancies to a configuration in $\mathrm{Dom}\,\mathrm{Tr}$ keeps it in $\mathrm{Dom}\,\mathrm{Tr}$, we may assume that all transitions involve at least one empty site.

Next, we exchange $-e_1$ and $-e_1 + e_2$:

$$M_2 = \mathrm{Ex}_2(-e_1 - le_2 + \mathcal{C}),$$

and move the mobile cluster back to $(l+2)e_1 + \mathcal{C}$.

$$M_3 = M_1^{-1}.$$

So far, we obtain a move $M_3 \circ M_2 \circ M_1$ with the associated permutation $(-e_1, -e_1 + e_2)$.

Next, we move the cluster, exchange $-e_1 + e_2$ with e_2 and the move it back:

$$M_4 = \mathrm{Tr}_{-1}((l+1)e_1 + e_2 + \mathcal{C}) \circ \mathrm{Tr}_{-1}((l+2)e_1 + e_2 + \mathcal{C}) \circ \mathrm{Tr}_2((l+2)e_1 + \mathcal{C}),$$
$$M_5 = \mathrm{Ex}_{-1}(le_1 + e_2 + \mathcal{C}),$$
$$M_6 = M_4^{-1}.$$

This results in a move $M_6 \circ M_5 \circ M_4$ associated to the permutation $(-e_1 + e_2, e_2)$.

In the same manner we construct a move M_7 associated with $(e_2, e_1 + e_2)$ and a move M_8 associated with $(e_1 + e_2, e_1)$.

We end up with the desired multistep move $\mathrm{Hop} = M_8 \circ M_7 \circ M_6 \circ M_5 \circ M_4 \circ M_3 \circ M_2 \circ M_1$. □

Claim 7.11. There exists a permutation σ_1 and a move M_{σ_1} such that:

(1) $\mathrm{Dom}\, M_{\sigma_1} = \left\{\eta : \eta(0) = 1 \text{ and } \hat{\mathcal{C}} \text{ is empty}\right\}$.
(2) M_{σ_1} is deterministic, compatible with σ_1.
(3) $\sigma_1(0) = e_1$ and $\sigma_1(\hat{\mathcal{C}}) = e_1 + \hat{\mathcal{C}}$.
(4) For all t, at least one of the two sites x_t or $x_t + e_t$ must be empty.

Proof. The move M_{σ_1} is given by

$$M_{\sigma_1} = \mathrm{Tr}_1((l+2)e_1 + \mathcal{C}) \circ \mathrm{Tr}_1((l+1)e_1 + \mathcal{C}) \circ \mathrm{Ex}_{-1}((l+1)e_1 + \mathcal{C}) \circ \mathrm{Tr}_{-1}((l+2)e_1 + \mathcal{C}) \circ \mathrm{Hop}.$$

□

So far, we constructed the permutation σ_1 defining the auxiliary model, and the move M_{z_0,σ_1} required in Hypothesis 6.3 (for $z_0 = 0$ hence for all z_0). This gives us automatically $\sigma_{-1} = \tau_{-e_1}\sigma_1^{-1}\tau_{e_1}$, and the move $M_{e_1,\sigma_{-1}} = M_{0,\sigma_1}^{-1}$, which provides $M_{z_0,\sigma_{-1}}$ for all z_0.

In order to propagate in other directions, we use the following claim:

Claim 7.12. For and $\alpha \in [1]$, there exists a permutation σ_α and a move M_{σ_α} such that:

(1) $\operatorname{Dom} M_{\sigma_\alpha} = \left\{\eta : \eta(0) = 1 \text{ and } \hat{\mathcal{C}} \text{ is empty}\right\}$.
(2) M_{σ_α} is deterministic, compatible with σ_α.
(3) $\sigma_\alpha(0) = e_\alpha$ and $\sigma_\alpha(\hat{\mathcal{C}}) = e_\alpha + \hat{\mathcal{C}}$.
(4) For all t, at least one of the two sites x_t or $x_t + e_t$ must be empty.

Proof. Claim 7.11 shows the case $\alpha = 1$.

The construction for $\alpha \neq 1$ is similar to the previous claims. Start by exchanging $-e_1$ with $-e_\alpha$ (in the exact same manner as the move $M_6 \circ M_5 \circ M_4 \circ M_3 \circ M_2 \circ M_1$ in the proof of Claim 7.10). Then translate the mobile cluster from $(l+2)e_1 + \mathcal{C}$ to $(l+2)e_\alpha + \mathcal{C}$. This brings us to the same setting as Claim 7.11, where the direction 1 is replaced by α. We may then use the same construction in order to move $\{0, -e_\alpha\} \cup ((l+2)e_\alpha + \mathcal{C})$ one step in the direction e_α. Finally, move the mobile cluster back from $(l+3)e_\alpha + \mathcal{C}$ to $(l+2)e_1 + e_\alpha + \mathcal{C}$ and the vacancy at 0 to $e_\alpha - e_1$. □

Theorem 7.1 then follows from Claim 7.12, Lemma 7.8, and Lemma 7.9. □

8. Questions

- The proofs given here show polynomial divergence of time scales as q tends to 0. Is it possible to identify the exact exponent of this divergence?
- What is the qualitative behavior of the different quantities described here when changing q? Are they continuous? Smooth? We expect them to be monotone (since decreasing q should "slow down" the system), but the nonattractivity of the model makes it difficult to prove.
- Variational formulas can also be used to approximate different quantities, and not just find bounds—consider, for example, the diffusion coefficient D. We may define, for $\Lambda \subset \mathbb{Z}^d$,

$$u \cdot D^{(\Lambda)} u = \frac{1}{2q(1-q)} \min_f \mu \left[\sum_{\alpha=1}^{d} c_{0,e_\alpha} \left(u \cdot e_\alpha (\eta(0) - \eta(e_\alpha)) + \sum_x \nabla_{0,e_\alpha} \tau_x f \right)^2 \right],$$

where the minimum is taken over functions $f : \{0,1\}^\Lambda \to \mathbb{R}$. Then $D = \lim_{\Lambda \to \mathbb{Z}^d} D^{(\Lambda)}$.

[1] evaluated this minimum, obtaining (nonrigorously) an approximate expression for D of the Kob-Andersen model, which is a cooperative kinetically constrained lattice gas. In their case, as q tends to 0, larger and larger boxes Λ must be taken in order to have a good approximation of D. We know that since any finite Λ gives $D^{(\Lambda)}$ polynomial in q, and for the Kob-Andersen model the diffusion coefficient decays superpolynamially.

In noncooperative models, the decays is polynomial, so one may hope that a finite box Λ could provide a good approximation of D for all q. For the model in Example 2.1 an empty

Λ already gives the correct diffusion coefficient up to a factor 2. What happens in other noncooperative models? Can we say that $D/D^{(\Lambda)} \to 1$ uniformly in q?

- Extend Theorem 5.6 to models satisfying Hypothesis 5.5 in all dimensions, or more generally to all noncooperative models.
- Given the positivity of the diffusion coefficient (Theorem 6.1), it is natural to conjecture convergence to the hydrodynamic limit of all noncooperative kinetically constrained lattice gases. Can we show it for models other than the one studied in [11]? Proving convergence for nongradient models (e.g. the model in Example 2.1) is an interesting (and challenging) problem.
- We expect the equilibrium fluctuations to converge to a Gaussian field (see, e.g., [27, II.2]), with the diffusion coefficient studied in Section 6. Can this be proven?
- Studying the diffusivity of cooperative kinetically constrained models. Results analogous to theorems 4.1, 6.1, and 7.1 have been shown for the Kob-Andersen model ([22, 25, 4, 9]). To the author's knowledge, other cooperative models have not been studied in the mathematical literature. Can one understand ergodicity properties of cooperative models? Does ergodicity always imply diffusivity? How do typical time scales diverge near criticality?

References

[1] Chikashi Arita, P.L. Krapivsky, and Kirone Mallick. Bulk diffusion in a kinetically constrained lattice gas. *Journal of Physics A: Mathematical and Theoretical*, 51(12):125002, 2018.

[2] Lorenzo Bertini and Cristina Toninelli. Exclusion processes with degenerate rates: convergence to equilibrium and tagged particle. *Journal of Statistical Physics*, 117(3):549–580, 2004.

[3] Oriane Blondel, Patrícia Gonçalves, and Marielle Simon. Convergence to the stochastic Burgers equation from a degenerate microscopic dynamics. *Electron. J. Probab.*, 21:Paper No. 69, 25, 2016.

[4] Oriane Blondel and Cristina Toninelli. Kinetically constrained lattice gases: tagged particle diffusion. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(4):2335–2348, 2018.

[5] Nicoletta Cancrini, Fabio Martinelli, Cyril Roberto, and Cristina Toninelli. Kinetically constrained spin models. *Probab. Theory Related Fields*, 140(3-4):459–504, 2008.

[6] Nicoletta Cancrini, Fabio Martinelli, Cyril Roberto, and Cristina Toninelli. Kinetically constrained lattice gases. *Comm. Math. Phys.*, 297(2):299–344, 2010.

[7] Persi Diaconis and Laurent Saloff-Coste. Comparison techniques for random walk on finite groups. *The Annals of Probability*, pages 2131–2156, 1993.

[8] Persi Diaconis and Mehrdad Shahshahani. Generating a random permutation with random transpositions. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 57(2):159–179, 1981.

[9] Anatole Ertul and Assaf Shapira. Self-diffusion coefficient in the Kob-Andersen model. *Electronic Communications in Probability*, 26:1–12, 2021.

[10] Juan P. Garrahan, Peter Sollich, and Cristina Toninelli. Kinetically constrained models. *Dynamical heterogeneities in glasses, colloids, and granular media*, 150:111–137, 2011.

[11] Patrícia Gonçalves, Claudio Landim, and Cristina Toninelli. Hydrodynamic limit for a particle system with degenerate rates. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(4):887–909, 2009.

[12] Ivailo Hartarsky. Refined universality for critical kcm: upper bounds. *arXiv preprint arXiv:2104.02329*, 2021.

[13] Ivailo Hartarsky and Laure Marêché. Refined universality for critical kcm: lower bounds. *Combinatorics, Probability and Computing*, 31(5):879–906, 2022.

[14] Ivailo Hartarsky, Laure Marêché, and Cristina Toninelli. Universality for critical kcm: infinite number of stable directions. *Probability Theory and Related Fields*, 178(1-2):289–326, 2020.

[15] Ivailo Hartarsky, Fabio Martinelli, and Cristina Toninelli. Universality for critical kcm: finite number of stable directions. *The Annals of Probability*, 49(5):2141–2174, 2021.

[16] C. Kipnis and S. R. S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.*, 104(1):1–19, 1986.

[17] Claude Kipnis and Claudio Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[18] Donald Ervin Knuth. *The art of computer programming*, volume 3: Sorting and searching. Addison-Wesley Reading, MA, 1998.

[19] Walter Kob and Hans C. Andersen. Kinetic lattice-gas model of cage effects in high-density liquids and a test of mode-coupling theory of the ideal-glass transition. *Physical Review E*, 48(6):4364, 1993.

[20] Laure Marêché, Fabio Martinelli, and Cristina Toninelli. Exact asymptotics for duarte and supercritical rooted kinetically constrained models. 2020.

[21] Fabio Martinelli, Robert Morris, and Cristina Toninelli. Universality results for kinetically constrained spin models in two dimensions. *Communications in mathematical physics*, 369(2):761–809, 2019.

[22] Fabio Martinelli, Assaf Shapira, and Cristina Toninelli. Diffusive scaling of the Kob-Andersen model in $\mathbb{Z}^d$. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 56(3):2189–2210, 2020.

[23] Yukio Nagahata. Lower bound estimate of the spectral gap for simple exclusion process with degenerate rates. *Electronic Journal of Probability*, 17:1–19, 2012.

[24] F. Ritort and P. Sollich. Glassy dynamics of kinetically constrained models. *Advances in Physics*, 52(4):219–342, 2003.

[25] Assaf Shapira. Hydrodynamic limit of the kob-andersen model. *arXiv preprint arXiv:2003.08495*, 2020.

[26] Herbert Spohn. Tracer diffusion in lattice gases. *J. Statist. Phys.*, 59(5-6):1227–1239, 1990.

[27] Herbert Spohn. *Large Scale Dynamics of Interacting Particles*. Springer-Verlag Berlin Heidelberg, 1991.

[28] Eial Teomy and Yair Shokef. Hydrodynamics in kinetically constrained lattice-gas models. *Physical Review E*, 95(2):022124, 2017.

[29] Cristina Toninelli, Giulio Biroli, and Daniel S. Fisher. Cooperative behavior of kinetically constrained lattice gas models of glassy dynamics. *J. Stat. Phys.*, 120(1-2):167–238, 2005.

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