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Relaxation Time and Topology in 1D $O(N)$ Models

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Abstract
We discuss the relaxation time (inverse spectral gap) of the one dimensional $O(N)$ model, for all N and with two types of boundary conditions. We see how its low temperature asymptotic behavior is affected by the topology. The combination of the space dimension, which here is always 1, the boundary condition (free or periodic), and the spin state $\mathbb{S}^{N-1}$, determines the existence or absence of non-trivial homotopy classes in some discrete version. Such non-trivial topology reflects in bottlenecks of the dynamics, creating metastable states that the system exits at exponential times; while when only one homotopy class exists the relaxation time depends polynomially on the temperature. We prove in the one dimensional case that, indeed, the relaxation time is a proxy to the model's topological properties via the exponential/polynomial dependence on the temperature.

Keywords Relaxation times · Topological effects · Metastability · Classical spin models

1 Introduction

The investigation of the low-temperature behavior of classical spin systems with continuous symmetry, such as the $O(N)$ model on a lattice, is a source of many fascinating questions in equilibrium statistical mechanics [7, 17]. For example, in the two dimensional XY model, a deep understanding of the interplay between the spin wave approximation and topological aspects such as vortex formation poses significant mathematical challenges, see [5, 8, 9, 13, 15] for some classical works, and see e.g. [10, 16] for more recent studies. To delve deeper into these questions, it is natural to study the Langevin dynamics associated to the $O(N)$ model, that is the reversible diffusion process with stationary distribution given by the $O(N)$

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Gibbs measure. In the mean-field case, a comprehensive analysis of the relaxation time, or inverse spectral gap, of Langevin dynamics for the $O(N)$ model, has been achieved recently in [2]. In particular, these results show that, when $N \geq 2$, in sharp contrast with the case of the Glauber dynamics for the Ising model ($N = 1$), the relaxation time of the mean field $O(N)$ model is at most linearly growing with the size of the system, at any fixed temperature. It is widely believed that such bounds should continue to hold for short range models as well, see e.g. [3]. In particular, it can be conjectured that for all lattice dimension $d \geq 2$, the $O(N)$ model on a lattice box with side L, for all $N \geq 2$, has relaxation time at most of order L^d at any fixed temperature, regardless of the boundary conditions. However, even establishing that relaxation times grow at most at a polynomial rate with L is a notoriously difficult open problem.

As a much more modest objective, in this note, we explore the presence of significant topological effects in the simpler one-dimensional setting. In a one-dimensional system, it is well known that the relaxation time is of order 1 at any fixed temperature. However, it was recently observed in [4] that in the one-dimensional XY model ($N = 2$), when periodic boundary conditions are imposed, as the inverse temperature β grows logarithmically with the size L of the chain, topologically induced metastable phases emerge, which correspond to distinct global winding numbers of the spin chain. When β increases as $\log L$, the free energy barrier between these states also increases linearly with β, leading to hitting times that are exponentially large in β. As we will see, this system exhibits relaxation times that grow exponentially with β. This phenomenon is specific of the periodic chain, and cannot occur for e.g. free boundary conditions. Indeed, it is a consequence of the fact that the global winding number of a periodic XY chain is a topological invariant, and two distinct phases cannot be connected by a homotopy, giving rise to a topological bottleneck. The main goal of this paper is to show that these topological effects on the dynamics are not present when $N \geq 3$. The point is that when $N \geq 3$, one can connect any two configurations of the spin chain by a continuous deformation. In particular, by using a continuous version of the canonical path method, we will show that, as a function of β, the relaxation time of the one-dimensional $O(N)$ model with $N \geq 3$ can grow at most polynomially in a periodic chain. We will also note that the same holds true for $O(N)$ models, this time for any $N \geq 2$, if one takes free boundary conditions instead.

1.1 Model and Results

Given integers $N \geq 2$, and $L \geq 2$, the one-dimensional $O(N)$ model of size L, with free or periodic boundary conditions is defined, respectively, by the Hamiltonians

$$H_L^{\mathrm{f}}(S) = -\sum_{i=1}^{L-1} S_i \cdot S_{i+1}, \qquad H_L^{\mathrm{per}}(S) = -\sum_{i=1}^{L} S_i \cdot S_{i+1}. \tag{1}$$

Here $S_i \in \mathbb{S}^{N-1}$ denotes the i-th spin, $S_i \cdot S_{i+1}$ is the usual scalar product for vectors in $\mathbb{R}^N$, and we set $S_{L+1} \equiv S_1$ to obtain periodic boundary conditions in $H_L^{\mathrm{per}}(S)$. We write ν for Lebesgue's measure on $\mathbb{S}^{N-1}$ and let ν_L denote the corresponding product measure on the space $\Omega_L = (\mathbb{S}^{N-1})^L$ of the spin chain configurations. Thus, the free and periodic boundary condition $O(N)$ Gibbs measure at inverse temperature $\beta > 0$ is given, respectively, by the probability measures on Ω_L defined as

$$\mu_{L,\beta}^{\mathrm{f}}(dS) = \frac{\exp\left(-\beta H_L^{\mathrm{f}}(S)\right)}{Z_{L,\beta}^{\mathrm{f}}}\,\nu_L(dS), \qquad \mu_{L,\beta}^{\mathrm{per}}(dS) = \frac{\exp\left(-\beta H_L^{\mathrm{per}}(S)\right)}{Z_{L,\beta}^{\mathrm{per}}}\,\nu_L(dS), \tag{2}$$

where the partition function is defined, respectively, by

$$Z_{L,\beta}^{\mathrm{f,per}} = \int_{\Omega_L} \exp\left(-\beta H_L^{\mathrm{f,per}}(S)\right) \nu_L(dS).$$

The Langevin dynamics is defined as the reversible diffusion process on Ω_L with infinitesimal generator

$$\mathcal{L}_{L,\beta}^{\mathrm{f,per}} = \sum_{i=1}^{L} \left(\frac{1}{\beta} D_i^2 - (D_i H_L^{\mathrm{f,per}}) \cdot D_i\right), \tag{3}$$

where D_i denotes the gradient $\nabla_{\mathbb{S}^{N-1}}$ on the unit sphere acting on the i-th spin S_i. The generator (3) defines a self-adjoint operator on $L^2(\Omega_L, \mu_{L,\beta}^{\mathrm{f}})$ and $L^2(\Omega_L, \mu_{L,\beta}^{\mathrm{per}})$, respectively. For any smooth function $f : \Omega_L \to \mathbb{R}$, the associated Dirichlet form is given by

$$\mathcal{D}_{L,\beta}^{\mathrm{f,per}}(f, f) = \frac{1}{\beta} \sum_{i=1}^{L} \int_{\Omega_L} \|D_i f(S)\|^2 \mu_{L,\beta}^{\mathrm{f,per}}(dS), \tag{4}$$

and $\|\cdot\|$ denotes the vector norm. The spectral gap is defined by the variational principle

$$\mathrm{gap}_{L,\beta}^{\mathrm{f,per}} = \inf_f \frac{\mathcal{D}_{L,\beta}^{\mathrm{f,per}}(f, f)}{\mathrm{Var}_{L,\beta}^{\mathrm{f,per}}(f)}, \tag{5}$$

where $\mathrm{Var}_{L,\beta}^{\mathrm{f,per}}(f) = \mu_{L,\beta}^{\mathrm{f,per}}(f^2) - \mu_{L,\beta}^{\mathrm{f,per}}(f)^2$ denotes the variance functional with respect to $\mu_{L,\beta}^{\mathrm{f,per}}$ and f ranges over all non-constant smooth functions on Ω_L. The relaxation time is defined as the inverse of the spectral gap

$$T_{\mathrm{rel}}^{\mathrm{f,per}}(L, \beta) = \frac{1}{\mathrm{gap}_{L,\beta}^{\mathrm{f,per}}}. \tag{6}$$

At any fixed β the one-dimensional nearest neighbor spin system satisfies exponential decay of covariances uniformly in the boundary conditions [7]. Then it is not difficult to prove, see e.g. [14], that the system has a uniformly positive spectral gap, that is there exists a constant $C(N, \beta)$ independent of L, such that

$$T_{\mathrm{rel}}^{\mathrm{f,per}}(L, \beta) \le C(N, \beta). \tag{7}$$

Here we are interested in detecting topologically induced slowdown effects on the relaxation to equilibrium which could appear in the case where β grows with L. In particular, as shown in [4], these phenomena do occur in the XY model ($N = 2$) in the case of periodic boundary conditions, when β is at least of order $\log L$.

We start by showing that in the case of free boundary conditions there is no topologically induced slowdown, in the sense that the relaxation time is upper bounded as follows.

Theorem 1.1 ($O(N)$ model with free boundary) *For any $N \ge 2$, $L \in \mathbb{N}$, and $\beta \ge 1$,*

$$T_{\mathrm{rel}}^{\mathrm{f}}(L, \beta) \le C(N)\, L^2 \beta^{(N+1)/2}, \tag{8}$$

where $C(N)$ depends only N. In particular, for each fixed N, it grows at most polynomially in β.

We note that, in light of the bound (7), the above estimate becomes relevant only when β grows with L. In this case, if β is at least $C \log L$ for some large C, as we discuss in Section 1.2 below, a diffusive scaling of the relaxation time should be expected, and thus at least qualitatively, the bound in Theorem 1.1 should be tight. In the case of periodic boundary conditions, in agreement with [4], we show the following bounds quantifying the topologically induced slowdown for $N = 2$. We note that, while [4] discuss the transition time between metastable states, we study the relaxation time (which is well-defined also in the absence of metastability); and while they describe similar phenomena one can not be directly deduced from the other.

Theorem 1.2 (XY model with periodic boundary) *Let $N = 2$. For $L \in \mathbb{N}$, and $\beta \geq 1$,*

$$T_{\mathrm{rel}}^{\mathrm{per}}(L, \beta) \leq C\beta^{3/2} e^{2\beta} L^2, \tag{9}$$

for some absolute constant C. Moreover, for any $L \in \mathbb{N}$, there exists a constant $c(L)$ depending on L such that for all $\beta \geq 1$,

$$T_{\mathrm{rel}}^{\mathrm{per}}(L, \beta) \geq c(L)\beta\, e^{(2-C_0 L^{-1})\beta}, \tag{10}$$

where C_0 is an absolute constant.

The lower bound (10) is based on a rather crude argument and provides no meaningful L-dependance. However, as discussed in Section 3.2, the leading exponential term $e^{2\beta}$ in (9)-(10) captures the correct metastable behavior associated to the energy barrier of size 2β between states with winding number zero and states with non-zero winding number. We refer to Remark 3.1 for the sketch of a finer energy-entropy argument providing quantitative L dependance in the metastable regime $\beta \geq C \log L$.

Finally, we prove that there is no topologically induced slowdown for $N \geq 3$.

Theorem 1.3 ($O(N)$ model on the cycle, $N \geq 3$) *For $N \geq 3$, $L \in \mathbb{N}$, and $\beta \geq 1$,*

$$T_{\mathrm{rel}}^{\mathrm{per}}(L, \beta) \leq C(L)\beta. \tag{11}$$

Moreover, in the case of the Heisenberg model ($N = 3$), one can take $C(L) = e^{CL\log L}$ for some constant C not depending on L and β.

1.2 Discussion, Conjectures and Open Problems

We emphasize that these estimates are far from optimal and do not capture all features of relaxation to equilibrium of the spin chain. However, they are sufficient to rule out the presence of topological bottlenecks in the relaxation process for all $N \geq 2$ in the case of free boundary and for $N \geq 3$ in the case of periodic boundary. Let us give some comments on our proofs. Roughly speaking, for Theorem 1.1 we use the fact that the spin chain has a product structure in the case of free boundary conditions, when one considers the "increment" variables $S_{i+1} - S_i$. This is achieved by a suitable change of variables that allows a convenient representation for the Hamiltonian. In this setup, a simple tensorization argument applies and one obtains the estimate (8) by changing back to the original spin variables. The explicit dependance on the variable β in (8) is obtained by a quantitative bound on the spectral gap for a single increment variable. The upper bound in (9) is obtained by reducing the problem to the free boundary case via a perturbation argument. On the other hand, for the lower bound in (9) we use an upper bound on the Cheeger constant. This is based on the choice of a suitable bottleneck event that was already analysed in [4].

The proof of Theorem 1.3 requires more work. We use a continuous version of the so-called canonical path method, see e.g. [18] for a classical formulation in the discrete setting. The main idea is to construct a path consisting of a continuous, energy decreasing transformation, which allows one to move any given configuration of L points on a sphere $\mathbb{S}^{N-1}$ to a configuration where all points lie in a small neighbourhood of a pole of the sphere. Once the system is confined to such a neighbourhood, convexity considerations allow us to conclude the desired statement. Checking that such a construction is possible, and controlling the entropy associated to the contracting path requires some non-trivial analysis. Note that this can only work in the case $N \geq 3$, since for $N = 2$ it is prevented by the topological obstruction discussed above.

Concerning dimension higher than one, in light of the above, it is natural to conjecture that if one considers the $O(N)$ model on the d-dimensional torus $(\mathbb{Z}/L\mathbb{Z})^d$, then topological bottlenecks are related to the homotopy structure. More precisely, one expects the relaxation time to grow exponentially with β when there are some non-trivial homotopy classes of maps from $\mathbb{T}^d \to \mathbb{S}^{N-1}$.

Indeed, when β is large the angle between two neighboring spins is small, and the discrete configuration of the spin system looks like a continuous field, i.e., a map from $\mathbb{T}^d$ to $\mathbb{S}^{N-1}$. With this in mind, the dynamics of the spin system corresponds to homotopy of the continuous field, and if there are several homotopy classes, then moving from one to the other requires the creation of a discontinuity. The energetic cost of this discontinuity creates the topological bottleneck.

While this description provides a good intuition to explain the results in this paper, in a more general setting we expect such metastability to depend also on the Riemannian structure of $\mathbb{S}^{N-1}$ and not solely its topology. Consider for example an hourglass spin state, which has the same topology as $\mathbb{S}^2$ but non-constant curvature. As in the $O(3)$ model, the ground state is when all spins point in the same direction. However, this system contains a metastable state, where spins are placed, equally spaced, on the narrow part of the hourglass. This narrow part looks like $\mathbb{S}^1$, and the metastable states will correspond to non-zero winding around it. In this example, the energetic cost is due to the continuous transformation and not the creation of discontinuities. We therefore see that endowing the same topological sphere with a Riemannian structure other than the standard one (i.e., the homogeneous metric induced by the scalar product in $\mathbb{R}^3$) could change the metastability properties.

This example suggests that homotopy classes of maps do not provide sufficient information to describe the metastability properties of the system. To remedy this, introduce the following energy functional. For M, M' two compact Riemannian manifolds, and $f : M \to M'$, let $\mathcal{E}(f)$ be the energy of f:

$$\mathcal{E}(f) = \int_M \|Df(x)\|^2_{f(x)} \mathrm{d}x, \tag{12}$$

where $\| \, \|_p$ is the norm on the tangent space of M' at p.

Extrema of the functional $\mathcal{E}$, called *harmonic maps*, are well studied mathematical objects, see e.g. [6]. Loosely stated, our general conjecture is that if one considers a model on some graph G with spin taking value in the manifold M', and if

- the graph G "approximates well" the manifold M (for example: the graph Laplacian on G is close in some sense to the Laplacian on M),
- the Hamiltonian of the model is a "approximate discrete version" of the energy functional $\mathcal{E}$. This is the generic behavior of nearest-neighbor attractive interaction near low energy states.

Then, the existence of metastable states is equivalent to the existence of non-trivial local minimizers of $\mathcal{E}$. Specializing to the spin $O(N)$ model, one obtains the next (more precise) conjecture.

Conjecture 1.4 *Let M be a compact Riemannian manifold, and define the energy functional operating on $f : M \to \mathbb{S}^{N-1}$ as*

$$\mathcal{E}(f) = \int_M \|Df(x)\|^2 \mathrm{d}x, \tag{13}$$

where the norm is taken with respect to the standard Riemannian metric on the sphere. Consider the $O(N)$ model on a graph G discretizing M. Then the relaxation time of the corresponding Langevin dynamics grows exponentially fast with β if and only if $\mathcal{E}$ has non-trivial (i.e., non-constant) local minima.

In particular, for the case of $\mathbb{Z}^d$ with free boundary conditions, there should be no such bottlenecks, that is relaxation times growing polynomially as a function of β, for all $N \geq 2$, since here the homotopy group is always trivial and the harmonic mappings are all constant. However, we leave it as an open problem to obtain quantitative bounds in dimension $d > 1$.

While we believe that for $M = \mathbb{S}^1$ or $M = (0, 1)$ our methods could be used in order to prove this conjecture, higher dimensional manifold will require much finer analysis. This is due to the underlying assumption, that the spin configuration can be approximated in the low energy regime by a continuous function.

An interesting example to study would be the case $N = 4$, $M = \mathbb{S}^3$: there are several homotopy classes of continuous maps from $\mathbb{S}^3$ to $\mathbb{S}^3$ (even countably many, by the Hurewicz theorem), but the energy $\mathcal{E}$ has no non-trivial local minima [6], so our conjecture is that there is no metastable behaviour in this case.

1.3 Scaling Limit in Low Temperature Heuristics

We conclude this introduction with a brief informal discussion of the behavior of the system for extremely low temperature, that is when β is large as a function of L, that could serve as a heuristic guide for a more precise analysis in this regime. For the sake of simplicity, we discuss the problem only for the XY model ($N = 2$) and give only a brief comment on the case $N \geq 3$ afterwards. When $N = 2$ we may parametrize S_i by a single angle $X_i \in [-\pi, \pi]$ and if $\beta = \beta(L)$ is very large we may assume there is a well defined lift to $\mathbb{R}$, so that our variables are now $X_i \in \mathbb{R}$, and the center of mass $\overline{X}_L = \frac{1}{L}\sum_{i=1}^{L} X_i$ satisfies the relation

$$\mathrm{d}\overline{X}(t) = \frac{\sqrt{2}}{\beta\sqrt{L}}\, dB(t)\,, \tag{14}$$

where $B(t)$ is a standard brownian motion. To see this, observe that by definition (3), the dynamics is given by the SDEs

$$\mathrm{d}X_i(t) = -\partial_{X_i} H(X)\,\mathrm{d}t + \frac{\sqrt{2}}{\beta} dB_i(t)\,, \qquad i = 1, \ldots, L \tag{15}$$

where the B_i's are independent standard Brownian motions, and the interaction has the form

$$H(X) = \sum_{i\sim j} h_{ij}\left(X_i - X_j\right) = \sum_{i\sim j} h_{ij}\left(X_j - X_i\right),$$

where the sum ranges over the edges of some finite graph, and $h_{ij} = h_{ji} = \cos(\cdot)$. The graph is the segment $\{1, \dots, L\}$ in the case of free boundary conditions, and it is the L-cycle in case of periodic boundary. Therefore,

$$\sum_i \partial_{X_i} H(X) = \sum_i \sum_{j:i\sim j} h'_{ij}(X_i - X_j) = -\sum_i \sum_{j:i\sim j} h'_{ij}(X_j - X_i) = 0.$$

In particular, (14) shows that the center of mass relaxes on a time scale proportional to L. Clearly, the above holds in any dimension $d \geq 1$, for the d-dimensional cube with side L with free or periodic boundaries, provided L is replaced by L^d. This can be seen as the starting point to establish volume order relaxation times estimates for the low temperature XY model. However, one has to keep in mind that this center of mass motion can be interpreted as a meaningful mode of the system, namely the global phase, only when all spins point in approximately the same direction. As noted in [4], for $d = 1$, the condition $\beta \gg L$ suffices to ensure that with large probability all spins are closely aligned, that is the winding number is zero, and there is a well defined lift X as above. When $\beta \not\gg L$, one needs to consider the sum of all spins, and of the corresponding Brownian motions in $\mathbb{R}^2$, as actual vectors, which makes the analysis considerably more involved; see [2] for a treatment of the mean field case.

Beyond the center of mass discussed above, one can also consider a stochastic PDE describing the continuum limit of the full configuration of the system. Consider the field $\phi : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ defined by

$$\phi(\xi, \tau) = \frac{1}{\sqrt{L}} X_{\lfloor L\xi/2\pi \rfloor}(L^2\tau). \tag{16}$$

When β is very large, we may approximate $\partial_{X_i} H(X) \approx X_i - X_{i-1} - (X_{i+1} - X_i) \approx (2\pi)^2 L^{-3/2} \Delta\phi(\xi, \tau)$, so, with $t = L^2\tau$ one has $L^{-1/2}\partial_{X_i} H(X)\mathrm{d}t \approx 4\pi^2\Delta\phi(\xi, \tau)\mathrm{d}\tau$. Moreover, reasoning as in [11, Section 2],

$$L^{-1/2}\frac{\mathrm{d}B_{\lfloor L\xi/2\pi \rfloor}(t)}{\mathrm{d}t} \approx W(\xi, \tau)\,,$$

where W is space-time white noise on $\mathbb{S}^1 \times \mathbb{R}$. In conclusion, from (15), the field ϕ satisfies the Edwards-Wilkinson equation, or stochastic heat equation

$$\partial_\tau \phi(\xi, \tau) \approx 4\pi^2\Delta\phi(\xi, \tau) + \sqrt{2}\beta^{-1} W(\xi, \tau). \tag{17}$$

We note that the continuum approximation discussed above should give a valid description of the system, on suitable time scales, provided β grows at least logarithmically with L. In particular, this suggests a diffusive time scale of order L^2 for relaxation, up to polylog corrections, when the system has free boundary conditions, as pointed out after Theorem 1.1. After this diffusive time scale, the system will relax by pure diffusion of the global phase. If we consider periodic boundary conditions, then the situation is different, since one has to impose the condition that $\phi(2\pi, \tau) - \phi(0, \tau) =$ winding number, and thus the equation (17) is valid only within a given homotopy class, i.e. it describes the system for times much smaller than the metastable times $T_{\mathrm{MS}} \approx e^{2\beta - \log(L)}$ detected in [4], at which the XY chain changes winding number. On the metastable time scale T_{MS} the dynamics will involve a random walk between the adjacent homotopy classes corresponding to ± 1 jumps of the global winding number.

The above heuristic analysis can be in principle repeated for any $N \geq 3$, by working with the local coordinates chosen to parametrize the sphere $\mathbb{S}^{N-1}$. One gets, independently for each coordinate, a SDE for the center of mass and a stochastic heat equation as above. However, because of the dependence on the choice of local coordinates on the manifold, the interpretation of these equations is no longer obvious in this case. We leave it as an open question to identify the limiting process.

2 Free Boundary Conditions

In this section we prove Theorem 1.1. We start by choosing appropriate coordinates.

2.1 Coordinates

For $N = 2$, we parametrize the system using angles $\theta_1, \ldots, \theta_L \in [0, 2\pi]$ and by setting

$$S_i = \big(\cos(\theta_1 + \cdots + \theta_i), \sin(\theta_1 + \cdots + \theta_i)\big).$$

The uniform measure on $(\mathbb{S}^1)^L$ is then the image of the uniform measure on $[0, 2\pi]^L$ by the above mapping.

For $N \geq 3$, a vector $s \in \mathbb{S}^{N-1}$ can be parametrized as

$$s = s(\theta, v) = \cos(\theta)\mathrm{e}_1 + \sin(\theta)v\,,$$

$(\mathrm{e}_1, \ldots, \mathrm{e}_N)$ is the canonical orthonormal basis of $\mathbb{R}^N$, $\theta \in [0, \pi]$, and v is a unit vector in the orthogonal complement of e_1. Then, sampling s uniformly on $\mathbb{S}^{N-1}$ is equivalent to sample v uniformly on $\{x : x \cdot \mathrm{e}_1 = 0, \|x\| = 1\}$, and θ proportionally to $\sin(\theta)^{N-2}$. For θ, v as before, denote $R_{v,\theta}$ the rotation matrix (in the standard basis) rotating the plane spanned by v and e_1 by an angle θ so that $R_{v,\theta}\mathrm{e}_1 = \cos(\theta)\mathrm{e}_1 + \sin(\theta)v$. In other words, $R_{v,\theta}$ is given by

$$R_{v,\theta}x = x + \big((\cos(\theta) - 1)x \cdot \mathrm{e}_1 - \sin(\theta)x \cdot v\big)\mathrm{e}_1 + \big(\sin(\theta)x \cdot \mathrm{e}_1 + (\cos(\theta) - 1)x \cdot v\big)v.$$

Let then $v_i, i = 1, \ldots, L$ be a sequence of uniform random variables on $\{x : x \cdot \mathrm{e}_1 = 0, \|x\| = 1\}$, let $\theta_i, i = 1, \ldots, L$ be an sequence of random variables on $[0, \pi]$ with density proportional to $\sin(\theta_i)^{N-2}$. Suppose $v_i, \theta_i, i = 1, \ldots, L$ forms an independent family. Set

$$R_i = R_{v_i,\theta_i}, \quad S_i = R_1 \ldots R_i \mathrm{e}_1.$$

The first ingredient we need is the next simple Lemma.

Lemma 2.1 *The sequence $S_i, i = 1, \ldots, L$ is an i.i.d. sequence of uniform random variables on $\mathbb{S}^{N-1}$.*

Proof It is sufficient to check that for every realization of $R_1, \ldots, R_{i-1}$, S_i is uniformly distributed on $\mathbb{S}^{N-1}$. This follows directly from rotation invariance of the spherical measure and the fact that $R_i\mathrm{e}_1$ is uniform on $\mathbb{S}^{N-1}$. □

The final ingredient we will need is a control over partial derivatives of S_i with respect to the angles θ_js and the vectors v_js. We note here that by $\frac{\partial}{\partial v_j} S_i$ we mean the differential with respect to v_j when fixing all other variables; in local coordinates it is given by the $(N-1) \times (N-1)$ Jacobian matrix, and we write $\Big\| \cdot \Big\|$ for the associated operator norm.

Lemma 2.2 *Let $N \geq 3$. For every $i, j \in \{1, \ldots, L\}$,*

$$\left\| \frac{\partial}{\partial \theta_j} S_i \right\| \leq 1, \quad \left\| \frac{\partial}{\partial v_j} S_i \right\| \leq 4. \tag{18}$$

Proof If $j > i$, both quantities are 0 and there is nothing to prove. Otherwise,

$$\left\| \frac{\partial}{\partial \theta_j} S_i \right\|^2 = \left\| \frac{\partial}{\partial \theta_j} R_j x \right\|^2 = (x \cdot \mathrm{e}_1)^2 + (x \cdot v)^2 \leq 1.$$

where $x = R_{j+1} \dots R_i \mathrm{e}_1$, and we used the fact that e_1, v are orthogonal and of norm 1. Then, for any $x \in \mathbb{R}^N$ and h of norm 1,

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon} (R_{v+\epsilon h, \theta} x - R_{v,\theta} x) =$$
$$= -\sin(\theta) x \cdot h \mathrm{e}_1 + (\cos(\theta) - 1) x \cdot h v + \big(\sin(\theta) x \cdot \mathrm{e}_1 + (\cos(\theta) - 1) x \cdot v \big) h,$$

so that for h of norm 1 in the tangent space of $\{y : \ y \cdot \mathrm{e}_1 = 0, \|y\| = 1\}$ at v_j, one has (as e_1, v_j, h are orthogonal)

$$\left\| \frac{\partial}{\partial v_j} S_i(\theta, v) \right\|^2 = \left\| \frac{\partial}{\partial v_j} R_{v_j, \theta_j} x \right\|^2 \leq 14$$

where again $x = R_{j+1} \dots R_i \mathrm{e}_1$. It follows that $\|\partial S_i / \partial v_j\| \leq \sqrt{14} \leq 4$. □

The interest of those parametrizations lies in the following identity: the Hamiltonian in (1) becomes

$$-H_L^{\mathrm{f}}(S(v, \theta)) = \sum_{i=1}^{L-1} (R_1 \dots R_{i+1} \mathrm{e}_1) \cdot (R_1 \dots R_i \mathrm{e}_1) = \sum_{i=1}^{L-1} (R_{i+1} \mathrm{e}_1) \cdot \mathrm{e}_1 = \sum_{i=2}^{L} \cos(\theta_i), \tag{19}$$

which gives a nice factorisation of the Boltzmann weight. The same identity holds for $N = 2$.

Therefore, we have that

$$\int_{\Omega_L} f(S) e^{-\beta H_L^{\mathrm{f}}(S)} \nu_L(dS) \propto \int d\theta dv \, f(S(\theta, v)) \sin(\theta_1)^{N-2} \prod_{i=2}^{L} \sin(\theta_i)^{N-2} e^{\beta \cos(\theta_i)},$$

where

- in the case $N \geq 3$, the right-hand-side integral is over $[0, \pi]^L \times (\mathbb{S}^{N-2})^L$, and we identified $\{x : \ \|x\| = 1, x \cdot \mathrm{e}_1 = 1\}$ with $\mathbb{S}^{N-2}$,
- in the case $N = 2$, the the right-hand-side integral is over $[0, 2\pi]^L$ and there is no variable v.

Next, for $N \geq 3$, we introduce the probability measures $\rho_1, \dots, \rho_L$ on $[0, \pi] \times \mathbb{S}^{N-2}$ given by

$$d\rho_1(\theta_1, v_1) \propto \nu(dv_1) \sin(\theta_1)^{N-2} d\theta_1,$$
$$d\rho_i(\theta_i, v_i) \propto \nu(dv_i) \sin(\theta_i)^{N-2} e^{\beta \cos(\theta_i)} d\theta_i, \ i = 2, \dots, L.$$

For $N = 2$, we use instead measures on $[0, 2\pi]$ given by

$$d\rho_1(\theta_1) \propto d\theta_1,$$
$$d\rho_i(\theta_i) \propto e^{\beta \cos(\theta_i)} d\theta_i, \ i = 2, \dots, L.$$

With this notation, we rewrite the expected value of some $f : \Omega_L \mapsto \mathbb{R}$ with respect to the Gibbs measure μ_L^{f} as

$$\mu_L^{\mathrm{f}}(f) = \otimes_{i=1}^{L} \rho_i \big(f(S(\theta, v)) \big). \tag{20}$$

2.2 Poincaré Inequalities for Increments Measures

For $f : [0, \pi] \times \{x : \|x\| = 1, x \cdot \mathrm{e}_1 = 0\} \to \mathbb{R}$, which maps (θ, v) to $f(\theta, v)$, denote as before $\frac{\partial}{\partial\theta}$ the partial derivative with respect to θ and $\frac{\partial}{\partial v}$ the partial derivative with respect to v (so that $\frac{\partial}{\partial v} f(\theta, v)$ is a linear function from the tangent space of $\{x : \|x\| = 1, x \cdot \mathrm{e}_1 = 0\}$ at v to $\mathbb{R}$). We also write $\mathrm{Var}_P(f)$ for the variance of f with respect to a probability measure P on $[0, \pi] \times \{x : \|x\| = 1, x \cdot \mathrm{e}_1 = 0\}$.

When $N \geq 3$, the measures ρ_i are product measures: the product of the uniform measure on $\{x : \|x\| = 1, x \cdot \mathrm{e}_1 = 0\} \equiv \mathbb{S}^{N-2}$ and of a measure on $[0, \pi]$. We start by proving Poincaré inequalities for these "elementary constituents" which in turn imply Poincaré inequalities for the ρ_is. We stress that we do not try to obtain the optimal constants, but we need a reasonable control over their dependency on the parameters.

We first prove the bounds used for $N \geq 3$.

Lemma 2.3 *Let $a, b \geq 0$. Let $P_{a,b}$ be the probability measure on $[0, \pi]$ with density proportional to $\sin(\theta)^a e^{b\cos(\theta)}$. Then, for any $f : [0, \pi] \to \mathbb{R}$ smooth,*

$$\mathrm{Var}_{P_{a,b}}(f) \leq c_1(a, b) E_{a,b}\big(|f'|^2\big),$$

where $E_{a,b}$ is the expectation with respect to $P_{a,b}$, and

$$c_1(a, b) = \frac{\pi^3 2^a b^{(a+1)/2}}{\int_0^{\sqrt{b}\pi/2} dx x^a e^{-x^2/2}} \text{ if } b > 0, \quad c_1(a, 0) = \frac{\pi^2 2(a+1)4^a}{\pi^a}.$$

Moreover, $c_1(a, b) \leq 8\pi^3 2^a b^{(a+1)/2}$ for $b \geq 1$.

Proof Start with $b > 0$. Let $P = P_{a,b}$. Set $C = \int_0^\pi dx \sin^a(x) e^{b\cos(x)}$. Then,

$$\begin{aligned}
\mathrm{Var}_P(f) &= \frac{1}{2C^2} \int_0^\pi dx \int_0^\pi dy (f(x) - f(y))^2 \sin^a(x) e^{b\cos(x)} \sin^a(y) e^{b\cos(y)} \\
&\leq \frac{\pi^2}{2C^2} \int_0^\pi dx \int_0^\pi dy \int_0^1 dt |f'(tx + (1-t)y)|^2 \sin^a(x) e^{b\cos(x)} \sin^a(y) e^{b\cos(y)}.
\end{aligned}$$

We can then use that on $[0, \pi]$, $g(x) = -a \ln(\sin(x))$ is convex and non-negative, therefore

$$g(x) + g(y) \geq g(tx + (1-t)y) + g((1-t)x + ty) \geq g(tx + (1-t)y),$$

and so $\sin^a(x) \sin^a(y) \leq \sin^a(tx + (1-t)y)$. Also,

$$\cos(x) + \cos(y) - \cos(tx + (1-t)y) \leq 1,$$

as cos is non-increasing on $[0, \pi]$ and less or equal to 1. Using these and changing variable to $z = tx + (1-t)y$, we obtain

$$\begin{aligned}
&\int_0^\pi dx \int_0^\pi dy \int_0^1 dt |f'(tx + (1-t)y)|^2 \sin^a(x) e^{b\cos(x)} \sin^a(y) e^{b\cos(y)} \\
&\leq e^b \int_0^1 dt \int_0^\pi \frac{dz}{t} \int_0^\pi dy |f'(z)|^2 \sin^a(z) e^{b\cos(z)} \mathbf{1}_{z-(1-t)y \in [0, t\pi]} \\
&\leq 2\pi e^b \int_0^\pi dz |f'(z)|^2 \sin^a(z) e^{b\cos(z)}
\end{aligned}$$

as $\int_0^1 dt \frac{1}{t} \int_0^\pi dy \mathbf{1}_{z-(1-t)y\in[0,t\pi]} \leq 2\pi$. Now,

$$C \geq \int_0^\pi dx \sin^a(x) e^{b-bx^2/2} \geq e^b \int_0^{\pi/2} dx \frac{x^a}{2^a} e^{-bx^2/2} = \frac{e^b}{2^a b^{(a+1)/2}} \int_0^{\sqrt{b}\pi/2} dx x^a e^{-x^2/2},$$

as $\cos(\gamma) \geq 1 - \gamma^2/2$ for $\gamma \in [0, \pi]$, and $\sin(\gamma) \geq \gamma/2$ for $\gamma \in [0, \pi/2]$. Combining all the estimates gives the main claim. The last point follows from (for $b \geq 1$)

$$\int_0^{\sqrt{b}\pi/2} dx x^a e^{-x^2/2} \geq \int_1^{\pi/2} dx e^{-x^2/2} \geq 1/8.$$

The case $b = 0$ follows the exact same path with the lower bound $C = \int_0^\pi \sin^a(x) \geq \frac{\pi^{a+1}}{2^{2a+1}(a+1)}$. □

The complete spectrum and eigenfunctions of the spherical Laplacian are known, and in particular its spectral gap is equal to $N - 1$ (see, e.g., [1, Section 2.2.3]).

Lemma 2.4 *Let $N \geq 2$. Let ν be the uniform probability measure on $\mathbb{S}^{N-1}$. Then, for any $f : \mathbb{S}^{N-1} \to \mathbb{R}$ smooth,*

$$\mathrm{Var}_\nu(f) \leq c_2(N) \nu\big(\|Df\|^2\big),$$

with $c_2(N) = \frac{1}{N-1}$.

The last bound is for the XY model case ($N = 2$).

Lemma 2.5 *Let $b \geq 0$. Let P_b be the probability measure on $[0, 2\pi]$ with density proportional to $e^{b\cos(\theta)}$. Then, for any smooth 2π-periodic $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathrm{Var}_{P_b}(f) \leq c_3(b) E_b\big(|f'|^2\big),$$

where E_b is the expectation with respect to P_b, and

$$c_3(b) = \frac{\pi^3\sqrt{b}}{\int_{-\sqrt{b}\pi}^{\sqrt{b}\pi} e^{-x^2/2}} \text{ if } b > 0, \quad c_3(0) = \frac{\pi^2}{2}.$$

Moreover, $c_3(b) \leq 2\pi^3\sqrt{b}$ for $b \geq 1$.

Proof Let $b > 0$, $C = \int_0^{2\pi} dx e^{b\cos(x)}$. One has

$$\begin{aligned}\mathrm{Var}_{P_b}(f) &= \frac{1}{2C^2} \int_0^{2\pi} dx \int_{-\pi}^{\pi} dy (f(x) - f(x+y))^2 e^{b(\cos(x)+\cos(x+y))} \\ &\leq \frac{\pi^2}{2C^2} \int_0^1 dt \int_0^{2\pi} dx \int_{-\pi}^{\pi} dy |f'(x+ty)|^2 e^{b(\cos(x)+\cos(x+y))}\end{aligned}$$

(we simply shifted the integration domain of y to the full period $[x - \pi, x + \pi]$ which preserves the integral by periodicity). We can then use

$$\cos(x) + \cos(x+y) - \cos(x+ty) \leq 1$$

as $y \in [-\pi, \pi]$, and proceed as in the proof of Lemma 2.3 to obtain

$$\mathrm{Var}_{P_b}(f) \leq \frac{\pi^2 e^b}{2C^2} \int_0^1 dt \int_0^{2\pi} dx \int_{-\pi}^{\pi} dy |f'(x+ty)|^2 e^{b\cos(x+ty)}$$

$$= \frac{2\pi^3 e^b}{2C^2} \int_0^{2\pi} dx |f'(x)|^2 e^{b\cos(x)} = \frac{\pi^3 e^b}{C} E_b(|f'|^2)$$

where we used periodicity in the second line. Now, as in the proof of Lemma 2.3, $C \geq \int_{-\pi}^{\pi} e^{b-bx^2/2} = \frac{e^b}{\sqrt{b}} \int_{-\sqrt{b}\pi}^{\sqrt{b}\pi} e^{-x^2/2}$, which gives the claim. A simplified version of the above treats the case $b = 0$ (there, $C = 2\pi$). □

We end this section by noticing that we have everything we need to control $\mu_L^{\mathrm{f}} = \otimes_{i=1}^L \rho_i$, as the ρ_is are themselves product of one or two of the above cases. More precisely, for $f : [0, \pi] \times \mathbb{S}^{N-2} \to \mathbb{R}$ smooth ($N \geq 3$), $i \geq 2$, and $(\theta, v) \sim \rho_i$, by the tensorization property of variance,

$$\begin{aligned}
\mathrm{Var}_{\rho_i}(f) &\leq P_{N-2,\beta} \otimes \nu\big(\mathrm{Var}_\nu(f(\theta, v)) + \mathrm{Var}_{P_{N-2,\beta}}(f(\theta, v))\big) \\
&\leq P_{N-2,\beta} \otimes \nu\big(c_2(N-2)\nu(\|\partial_v f(\theta, v)\|^2) + c_1(N-2, \beta) E_{N-2,\beta}(|\partial_\theta f(\theta, v)|^2)\big) \\
&= c_2(N-2)\rho_i\big(\|\partial_v f(\theta, v)\|^2\big) + c_1(N-2, \beta)\rho_i\big(|\partial_\theta f(\theta, v)|^2\big),
\end{aligned} \tag{21}$$

where $P_{a,b}$ is the measure of Lemma 2.3. A similar bound holds for $i = 1$.

2.3 Proof of Theorem 1.1

Introduce

$$c(\beta, N) = \begin{cases} \max(c_3(\beta), c_3(0)) & \text{if } N = 2, \\ \max(c_1(N-2, \beta), c_1(N-2, 0), c_2(N-2)) & \text{if } N \geq 3. \end{cases} \tag{22}$$

By Lemmas 2.3, 2.4, and 2.5, for β larger than 1,

$$c(\beta, N) \leq \pi^3 2^{N+1} \beta^{(N-1)/2}. \tag{23}$$

The upper bound in Theorem 1.1 is a consequence of the following Lemma and of (23).

Lemma 2.6 *Let $f : \Omega_L \to \mathbb{R}$ be smooth. Then,*

$$\mathrm{Var}_{\mu_L^{\mathrm{f}}}(f) \leq \beta\, 17 c(\beta, N) L^2 \mathcal{D}_{L,\beta}^{\mathrm{f}}(f, f),$$

where $c(\beta, N)$ is given by (22).

Proof Since μ_L^{f} is the product measure (20), we have that by the tensorization of variance,

$$\mathrm{Var}_{\mu_L^{\mathrm{f}}}(f) \leq \otimes_{i=1}^L \rho_i \Big(\sum_{i=1}^L \mathrm{Var}_{\rho_i}\big(f(\theta_1, v_1, \ldots, \theta_L, v_L)\big)\Big).$$

Applying (21), one obtains

$$\mathrm{Var}_{\mu_L^{\mathrm{f}}}(f) \leq c(\beta, N) \otimes_{i=1}^L \rho_i \Big(\sum_{i=1}^L \big(\|\partial_{v_i} f\|^2 + |\partial_{\theta_i} f|^2\big)\Big). \tag{24}$$

Moreover, by the chain rule, for $f : (\mathbb{S}^{N-1})^L \to \mathbb{R}$ smooth, one has

$$\partial_{v_i} f\big(S(\theta, v)\big) = \sum_{j=i}^L D_j f(S(\theta)) \cdot \partial_{v_i} S_j(\theta_1, v_1, \ldots, \theta_j, v_j).$$

and

$$\partial_{\theta_i} f\big(S(\theta, v)\big) = \sum_{j=i}^{L} D_j f(S(\theta)) \cdot \partial_{\theta_i} S_j(\theta_1, v_1, \ldots, \theta_j, v_j).$$

Observe that by our choice of coordinates (Lemma 2.2), for any $L \geq j \geq i \geq 1$,

$$\|\partial_{\theta_i} S_j(\theta_1, v_1, \ldots, \theta_L, v_L)\| \leq 1, \quad \|\partial_{v_i} S_j(\theta_1, \ldots, \theta_L, v_L)\| \leq 4.$$

Therefore, by Cauchy-Schwarz,

$$\begin{aligned}\sum_{i=1}^{L} \left(\|\partial_{v_i} f\|^2 + |\partial_{\theta_i} f|^2\right) &\leq \sum_{i=1}^{L} \left(16L \sum_{j=i}^{L} \|D_j f\|^2 + L \sum_{j=i}^{L} \|D_j f\|^2\right) \\ &\leq 17L^2 \sum_{i=1}^{L} \|D_i f\|^2.\end{aligned}$$

Plugging this in (24), and recalling (4) concludes the proof. □

3 *XY* Model on the Cycle

In this section we prove Theorem 1.2. We start with the proof of the upper bound.

3.1 Upper Bound

The proof is based on the upper bound for free boundary conditions in Theorem 1.1 and simple comparison between the free and the periodic boundary condition system. The first observation is that from the definitions (1)-(2) and the fact that $|S_i \cdot S_{i+1}| \leq 1$ it follows that the relative densities $\mathrm{d}\mu_L^{\mathrm{f}}/\mathrm{d}\mu_L^{\mathrm{per}}$, $\mathrm{d}\mu_L^{\mathrm{per}}/\mathrm{d}\mu_L^{\mathrm{f}}$ satisfy

$$\left\|\frac{\mathrm{d}\mu_L^{\mathrm{f}}}{\mathrm{d}\mu_L^{\mathrm{per}}}\right\|_\infty \leq \frac{Z_{L,\beta}^{\mathrm{per}}}{Z_{L,\beta}^{\mathrm{f}}} \exp(\beta), \qquad \left\|\frac{\mathrm{d}\mu_L^{\mathrm{per}}}{\mathrm{d}\mu_L^{\mathrm{f}}}\right\|_\infty \leq \frac{Z_{L,\beta}^{\mathrm{f}}}{Z_{L,\beta}^{\mathrm{per}}} \exp(\beta). \tag{25}$$

At this point, for any smooth function $f : \Omega_L \to \mathbb{R}$ one has

$$\begin{aligned}\mathrm{Var}_{\mu_L^{\mathrm{per}}}(f) &= \inf_{c\in\mathbb{R}} \mu_L^{\mathrm{per}}((f-c)^2) \leq \mu_L^{\mathrm{per}}((f-\mu_L^{\mathrm{f}}(f))^2) \\ &\leq \frac{Z_{L,\beta}^{\mathrm{f}}}{Z_{L,\beta}^{\mathrm{per}}} \exp(\beta)\, \mu_L^{\mathrm{f}}((f-\mu_L^{\mathrm{f}}(f))^2) = \frac{Z_{L,\beta}^{\mathrm{f}}}{Z_{L,\beta}^{\mathrm{per}}} \exp(\beta)\, \mathrm{Var}_{\mu_L^{\mathrm{f}}}(f).\end{aligned}$$

Similarly,

$$\mathcal{D}_{L,\beta}^{\mathrm{f}}(f, f) \leq \frac{Z_{L,\beta}^{\mathrm{per}}}{Z_{L,\beta}^{\mathrm{f}}} \exp(\beta)\, \mathcal{D}_{L,\beta}^{\mathrm{per}}(f, f).$$

Therefore, from Theorem 1.1 it follows that

$$\mathrm{Var}_{\mu_L^{\mathrm{per}}}(f) \leq e^{2\beta} C(2) L^2 \beta^{3/2} \mathcal{D}_{L,\beta}^{\mathrm{per}}(f, f).$$

This implies the upper bound (9).

3.2 Lower Bound

We use the variational principle (5). In order to construct an appropriate test function, we follow [4], where a proxy for the winding number of the spin chain was defined as follows. Let $S_i \in \mathbb{S}^1$ denote the i-th spin, with $S_{L+1} = S_1$, and write $[\theta]$ for the representative in the interval $(-\pi, \pi]$ of any $\theta \in \mathbb{S}^1$. If $S \in (\mathbb{S}^1)^L$ is such that $S_{i+1} - S_i \in \mathbb{S}^1 \setminus \{\pi\}$ for all $i = 1, \ldots, L$, define the function

$$\mathcal{W}(S) = \frac{1}{2\pi} \sum_{i=1}^{L} [S_{i+1} - S_i]. \tag{26}$$

Because of the periodic boundary condition, $\mathcal{W}$ is an integer, which can be interpreted as the winding number of the spin configuration. Moreover, the function $\mathcal{W}$ is continuous in its domain of definition D given by

$$D = \left\{ S \in (\mathbb{S}^1)^L : \; S_{i+1} - S_i \in \mathbb{S}^1 \setminus \{\pi\}, \; i = 1, \ldots, L \right\}. \tag{27}$$

Next define the events

$$B = \{S \in D : \; \mathcal{W}(S) = 0\}, \tag{28}$$

$$A_\delta = \{S \in D : \; \mathcal{W}(S) = 0 \; \text{ and } S_i - S_{i+1} \in [\pi - \delta, \pi + \delta] \text{ for some } i\}. \tag{29}$$

We let $f : (\mathbb{S}^1)^L \mapsto \mathbb{R}$ denote a $\mathcal{C}^\infty$ function such that $f = 0$ on B^c, $f = 1$ on $B \setminus A_\delta$ and such that $\|D_i f\|_\infty \le C_\delta$, where $C_\delta = O(1/\delta)$ is a constant independent of β, L. Since $D_i f(S) \neq 0$ implies that $S \in A_\delta$, using this function f in the variational principle (5) one obtains

$$\mathrm{gap}^{\mathrm{per}}_{L,\beta} \le \frac{1}{\beta} C_\delta^2 L \frac{\mu^{\mathrm{per}}_{L,\beta}(A_\delta)}{\mathrm{Var}^{\mathrm{per}}_{L,\beta} f}. \tag{30}$$

Moreover, by definition of f one has

$$\mathrm{Var}^{\mathrm{per}}_{L,\beta} f \ge \mu^{\mathrm{per}}_{L,\beta}(B \setminus A_\delta) \mu^{\mathrm{per}}_{L,\beta}(B^c). \tag{31}$$

It remains to estimate $\mu^{\mathrm{per}}_{L,\beta}(A_\delta)$ and the probabilities in (31).

For $\delta > 0$, define the events

$$B^0_\delta = \left\{ S_i - S^0_i \in [-\delta, \delta] \text{ for all } i = 1, \ldots, L \right\},$$
$$B^1_\delta = \left\{ S_i - S^1_i \in [-\delta, \delta] \text{ for all } i = 1, \ldots, L \right\},$$

where the configurations S^0, S^1, seen as variables in the complex plane, are defined by

$$S^0_j = 1, \qquad S^1_j = e^{2\pi i \frac{j}{L}}, \qquad j \in \{1, \ldots L\}.$$

Observe that, for small enough $\delta = \delta(L)$ depending on L,

$$B^0_\delta \in \{\mathcal{W} = 0\} \text{ and } B^1_\delta \in \{\mathcal{W} = 1\},$$

and, writing $H(S) := H^{\mathrm{per}}_L(S)$,

$$H(S) \ge 2 - \delta^2 - L, \qquad S \in A_\delta,$$
$$H(S) \le -(1 - \delta^2)L, \qquad S \in B^0_\delta,$$
$$H(S) \le -L + C(\delta + L^{-1}) + C\delta^2 L, \qquad S \in B^1_\delta,$$

for some absolute constant $C > 0$. With the notation $\mathrm{vol}(A) = \int_A \mathrm{d}S$ for any $A \subset \Omega$, $\Omega = \Omega_L$, we obtain

$$\mathrm{vol}(B_\delta^0)e^{\beta(1-\delta^2)L} \leq \int_\Omega e^{-\beta H(S)}\mathrm{d}S \leq \mathrm{vol}(\Omega)e^{\beta L}.$$

Therefore, writing $\mu = \mu_{L,\beta}^{\mathrm{per}}$,

$$\begin{aligned}\mu(A_\delta) = \frac{\int_{A_\delta} e^{-\beta H(S)}\mathrm{d}S}{\int_\Omega e^{-\beta H(S)}\mathrm{d}S} &\leq \frac{\mathrm{vol}(A_\delta)e^{-\beta(2-\delta^2-L)}}{\mathrm{vol}(B_\delta^0)e^{\beta(1-\delta^2)L}} \\ &= \frac{\mathrm{vol}(A_\delta)}{\mathrm{vol}(B_\delta^0)}\, e^{-\beta\left(2-\delta^2(L+1)\right)} = C_{\delta,L}\, e^{-\beta\left(2-\delta^2(L+1)\right)},\end{aligned}$$

where $C_{\delta,L}$ is a constant depending on δ, L. We can also estimate

$$\mu(B_\delta^0) = \frac{\int_{B_\delta^0} e^{-\beta H(S)}\mathrm{d}S}{\int_\Omega e^{-\beta H(S)}\mathrm{d}S} \geq \frac{\mathrm{vol}(B_\delta^0)e^{\beta(1-\delta^2)L}}{\mathrm{vol}(\Omega)e^{\beta L}} = c_{\delta,L}e^{-\delta^2 L\beta},$$

where $c_{\delta,L}$ is another constant depending on δ, L. Finally,

$$\mu(B_\delta^1) = \frac{\int_{B_\delta^1} e^{-\beta H(S)}\mathrm{d}S}{\int_\Omega e^{-\beta H(S)}\mathrm{d}S} \geq \frac{\mathrm{vol}(B_\delta^1)e^{\beta\left(L-C(\delta+L^{-1})-C\delta^2 L\right)}}{\mathrm{vol}(\Omega)e^{\beta L}},$$

which implies

$$\mu(B_\delta^1) \geq c_{\delta,L}e^{-\left(C(\delta+L^{-1})+C\delta^2 L\right)\beta},$$

for some other constant $c_{\delta,L}$ depending on δ, L. Summarizing, we obtain the following estimate.

Proposition 3.1 *For all $L \in \mathbb{N}$, $\delta = \delta(L) \leq L^{-1}$, for all $\beta \geq 1$,*

$$\frac{\mu(A_\delta)}{\mu_{L,\beta}^{\mathrm{per}}(B \setminus A_\delta)\mu_{L,\beta}^{\mathrm{per}}(B^c)} \leq \frac{\mu(A_\delta)}{\mu(B_\delta^0)\mu(B_\delta^1)} \leq C_{\delta,L}e^{-(2-C_0L^{-1})\beta}, \tag{32}$$

where $C_{\delta,L}$ is a constant depending on δ, L, and C_0 is an absolute constant.

The lower bound in (10) now follows from (30)-(31) and Proposition 3.1.

Remark 3.1 (A finer lower bound for $\beta \geq C \log L$) The estimate in Proposition 3.1 captures the correct energy barrier of size 2β up to $O(L^{-1})$ corrections, see the matching upper bound (9). However, it provides no quantitative estimate in terms of the system size L. To resolve this, we observe that Proposition 3.1 can be considerably refined by adapting the analysis from [4]. Rather than giving an explicit derivation, we content ourselves with the following observations. One can use the arguments in [4, Section 4] to prove that for all L and $\beta \geq 1$,

$$\mu_{L,\beta}^{\mathrm{per}}(B) \geq \frac{1}{L}, \qquad \mu_{L,\beta}^{\mathrm{per}}(B^c) \geq \frac{1}{C_0\sqrt{\beta}L}e^{-C_0\beta/L},$$

where C_0 is an absolute constant, and that for any $\delta \in (0, 1)$ one has

$$\mu_{L,\beta}^{\mathrm{per}}(A_\delta) \leq C_0\beta L\delta e^{-\beta(2-\delta^2/2)}.$$

From these bounds it is not difficult to check that for an appropriate absolute constant C_0, if $\beta \geq C_0 \log L$, $L \geq C_0$, then for all fixed $\delta \in (0, 1)$, the left hand side of (32) is bounded from above by

$$C_0 \beta^{3/2} L^3 \delta \, e^{-\beta(2-\delta^2/2 - C_0 L^{-1})}.$$

Thus, from (30) and (31) one arrives at the following relaxation time lower bound: for all $\delta \in (0, 1)$, there exists a constant $c_\delta > 0$ such that for $L \geq 1/c_\delta$, and $\beta \geq C_0 \log L$, for an appropriate absolute constant C_0, one has

$$T^{\text{per}}_{\text{rel}}(L, \beta) \geq c_\delta \beta^{-1/2} L^{-4} \, e^{(2-\delta^2)\beta} \,. \tag{33}$$

4 $O(N)$ on the Cycle

Here we prove Theorem 1.3. In order to simplify notation we omit some subscripts and superscripts; we only discuss in this section the relaxation time for fixed $N \geq 3$, and fixed L and β, with periodic boundary conditions.

4.1 General Strategy

In order to bound the relaxation time from above, we will show how to bring a configuration, using a canonical path method, to a small neighborhood where the hamiltonian in convex.

Definition 4.1 The *arctic* is defined as the set of configurations where all spins are in the ball of radius $\arccos(0.99) \approx 0.02 \times 2\pi$ around e_1:

$$A = \{S \in \mathbb{S}^{N-1} : S \cdot \mathrm{e}_1 > 0.99\}^L. \tag{34}$$

The following is a simple consequence of our definitions.

Fact 1 *H^{per}_L is convex on A.*

Definition 4.2 Fix an open set $U \subseteq \Omega$. A path from U is a continuous function $\Phi : U \times [0, 1] \to \Omega$. We define the path's *energy* and *entropy* respectively as

$$\Delta H_\Phi(S, t) = H^{\text{per}}_L(\Phi(S, t)) - H^{\text{per}}_L(S), \tag{35}$$

$$\Delta \operatorname{Ent}_\Phi(S, t) = \log \left| \frac{d(\Phi(\cdot, t))_\# \nu_L}{d\nu_L} \right|, \tag{36}$$

where $\left| \frac{d(\Phi(\cdot,t))_\# \nu_L}{d\nu_L} \right|$ is the density of the push-forward measure of ν_L with respect to ν_L; in local coordinates it is given by the determinant of the Jacobian matrix. The *free energy barrier* of the path is defined by

$$\Delta F[\Phi] = \sup_{t \in [0,1]} \sup_{S \in U} \Delta H_\Phi(S, t) - \beta^{-1} \Delta \operatorname{Ent}_\Phi(S, t). \tag{37}$$

The *speed* of the path is defined by

$$v[\Phi] = \sup_{t \in [0,1]} \sup_{S \in U} \|\partial_t \Phi(S, t)\|, \tag{38}$$

where $\| \cdot \|$ denotes the euclidean norm

$$\|\xi\|^2 = \sum_{j=1}^{L} \|\xi_j\|^2 \,, \qquad \xi = (\xi_1, \ldots, \xi_L) \in \mathbb{R}^L.$$

The paths to be considered below are almost everywhere differentiable, so the above is well defined as an essential supremum.

Lemma 4.1 *Let $U_1, \dots, U_K$ be some finite collection of open subsets of Ω, and $\Phi_1, \dots, \Phi_K$ a collection of paths, $\Phi_i : U_i \times [0,1] \to \Omega$. Assume:*
(1) $\bigcup_{i=1}^K U_i = \Omega$.
(2) $\Phi_i(S,0) = S$ and $\Phi_i(S,1) \in A$ for all $S \in U_i$.
(3) $\Delta F := \sup_i \Delta F[\Phi_i]$ and $v := \sup_i v[\Phi_i]$ are both finite.
Then

$$T_{\mathrm{rel}} \leq 3K^2 \beta (v^2 e^{\beta \Delta F} + e^{2\beta\Delta F} \mu(A)) \tag{39}$$

In particular, if the energy barrier $\sup_i \sup_{t\in[0,1]} \sup_{S\in U_i} \Delta H_{\Phi_i}(S,t)$ is non-positive then the β-dependence of the relaxation time grows at most linearly.

Proof Fix some test function $f : \Omega \to \mathbb{R}$, and use the notation $\Phi_i(X) = \Phi_i(X,1)$. The variance of f with respect to $\mu := \mu_{L,\beta}^{\mathrm{per}}$ satisfies

$$\begin{aligned}
2\mathrm{Var}_{L,\beta}^{\mathrm{per}}(f) &= \int d\mu(X) d\mu(Y) (f(X) - f(Y))^2 \\
&\leq \sum_{i=1}^K \sum_{j=1}^K \int d\mu(X) d\mu(Y) \mathbb{1}_{U_i}(X) \mathbb{1}_{U_j}(Y) (f(X) - f(Y))^2 \\
&= \sum_{i=1}^K \sum_{j=1}^K \int d\mu(X) d\mu(Y) \mathbb{1}_{U_i}(X) \mathbb{1}_{U_j}(Y) \\
&\times (f(X) - f(\Phi_i(X)) + f(\Phi_i(X)) - f(\Phi_j(Y)) + f(\Phi_j(Y)) - f(Y))^2 \\
&\leq 3 \sum_{i=1}^K \sum_{j=1}^K \int d\mu(X) d\mu(Y) \mathbb{1}_{U_i}(X) \mathbb{1}_{U_j}(Y) (f(X) - f(\Phi_i(X)))^2 \\
&\quad + 3 \sum_{i=1}^K \sum_{j=1}^K \int d\mu(X) d\mu(Y) \mathbb{1}_{U_i}(X) \mathbb{1}_{U_j}(Y) (f(\Phi_i(X)) - f(\Phi_j(Y)))^2 \\
&\quad + 3 \sum_{i=1}^K \sum_{j=1}^K \int d\mu(X) d\mu(Y) \mathbb{1}_{U_i}(X) \mathbb{1}_{U_j}(Y) (f(\Phi_j(Y)) - f(Y))^2 \\
&\leq 6K \sum_{i=1}^K \int d\mu(X) \mathbb{1}_{U_i}(X) (f(X) - f(\Phi_i(X)))^2 \\
&\quad + 3 \sum_{i=1}^K \sum_{j=1}^K \int d\mu(X) d\mu(Y) \mathbb{1}_{U_i}(X) \mathbb{1}_{U_j}(Y) (f(\Phi_i(X)) - f(\Phi_j(Y)))^2 \\
&= 6K \sum_{i=1}^K (\mathrm{I})_i + 3 \sum_{i=1}^K \sum_{j=1}^K (\mathrm{II})_{i,j}
\end{aligned} \tag{40}$$

We start with the first term. For any i and $X \in \Omega$,

$$(f(X) - f(\Phi_i(X)))^2 = \left(\int_0^1 dt \sum_{j=1}^L D_j f(\Phi_i(X,t)) \cdot (\partial_t \Phi_i(X,t))_j \right)^2 \tag{41}$$

$$\leq v^2 \int_0^1 dt \sum_{j=1}^{L} \|D_j f(\Phi_i(X,t))\|^2, \tag{42}$$

where D_j denotes the gradient on the sphere $\mathbb{S}^{N-1}$ acting on the j-th spin. Plugging this into (I), recalling the definition of μ, and then changing variable to $X' = \Phi_I(X,t)$, yield

$$\text{(I)}_i \leq v^2 \int_0^1 dt \int d\mu(X) \mathbb{1}_{U_i}(X) \sum_{j=1}^{L} \|D_j f(\Phi_i(X,t))\|^2 \tag{43}$$

$$= \frac{v^2}{Z} \int_0^1 dt \int d\nu(X) e^{-\beta H(\Phi_i(X,t)) + \beta \Delta H_{\Phi_i}(X,t)} \mathbb{1}_{U_i}(X) \sum_{j=1}^{L} \|D_j f(\Phi_i(X,t))\|^2 \tag{44}$$

$$= \frac{v^2}{Z} \int_0^1 dt \int d\nu(X') e^{-\Delta\,\mathrm{Ent}_{\Phi_i}(X,t)} e^{-\beta H(X')} e^{\beta \Delta H_{\Phi_i}(X,t)} \mathbb{1}_{U_i}(X) \sum_{j=1}^{L} \|D_j f(X')\|^2 \tag{45}$$

$$\leq v^2 e^{\beta \Delta F} \int_0^1 dt \int d\mu(X') \sum_{j=1}^{L} \|D_j f(X')\|^2 = v^2 e^{\beta \Delta F} \sum_{j=1}^{L} \mu(\|D_j f\|^2) \tag{46}$$

$$= \beta v^2 e^{\beta \Delta F} \mathcal{D}^{\text{per}}_{L,\beta}(f,f). \tag{47}$$

For the second term, the change of variables $X' = \Phi_i(X,t)$, $Y' = \Phi_j(Y,t)$ leads using the same calculation to

$$\text{(II)}_{i,j} \leq e^{2\beta \Delta F} \int d\mu(X') d\mu(Y') \mathbb{1}_A(X') \mathbb{1}_A(Y') (f(X') - f(Y'))^2. \tag{48}$$

This last integral equals $\mu(A)^2$ times the variance of f under the conditional measure $\mu(\cdot|A)$. Since on A the Hamiltonian is convex, and since Ω has positive curvature 1, the Brascamp-Lieb inequality for $\mu(\cdot|A)$, see e.g. [12, Theorem 1.2], tells us that:

$$\frac{1}{2} \int d\mu(X|A) d\mu(Y|A) (f(X) - f(Y))^2 \leq \sum_{j=1}^{L} \mu(\|D_j f\|^2 | A). \tag{49}$$

We conclude that

$$\text{(II)}_{i,j} \leq 2e^{2\beta \Delta F} \mu(A) \sum_{j=1}^{L} \mu(\|D_j f\|^2) = 2\beta e^{2\beta \Delta F} \mu(A) \mathcal{D}^{\text{per}}_{L,\beta}(f,f). \tag{50}$$

Together with the bound (47), this ends the proof of the lemma. □

4.2 Constructing The Path: a Soft Argument for Any $N \geq 3$

We will construct paths to be used in Lemma 4.1 in three parts: first, we use the deterministic flow in order to bring the configuration near a critical point, where all spins are on a single *great circle*, defined as the intersection of the sphere with a 2 dimensional plane. Then, we pull the spins to a point perpendicular to that great circle. Finally, we rotate the configuration to the arctic. See Figure 1.

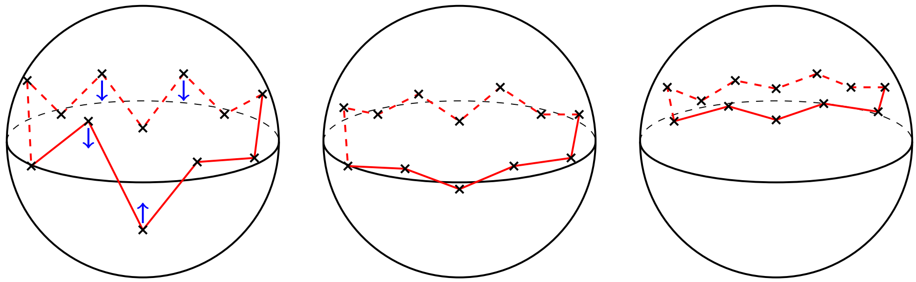

Fig. 1 Illustration of the path Ψ. In the first step all spins are brought to the vicinity of a great circle. Once there, we pull all spins towards a single point

Step 1: Approaching a Great Circle

Define the deterministic flow $\varphi : \Omega \times \mathbb{R} \to \Omega$ as the solution, for each $S \in \Omega$, of the differential equation

$$\frac{\partial \varphi(S,t)}{\partial t} = -DH(\varphi(S,t)), \tag{51}$$

$$\varphi(S,0) = S, \tag{52}$$

where we use $D = (D_j)_{j=1,\dots,L}$ for the L-vector of sphere gradients D_j.

Claim 1 *Let S be a stationary point of the flow. Then the spins $S_1, \dots, S_L$ all belong to the same great circle.*

Proof By explicit calculation in $\mathbb{R}^N$,

$$\begin{aligned} -D_i H(S) &= D_i \left[S_{i-1} \cdot S_i + S_i \cdot S_{i+1} \right] = S_{i-1} - (S_{i-1} \cdot S_i) S_i + S_{i+1} - (S_{i+1} \cdot S_i) S_i \\ &= S_{i-1} + S_{i+1} - (S_{i-1} \cdot S_i + S_{i+1} \cdot S_i) S_i . \end{aligned}$$

Therefore, $D_i H(S) = 0$ implies that S_{i+1} is in the linear span of S_{i-1} and S_i. It follows that if $DH(S) = 0$, then this holds for all i, and therefore all the spins belong to the same 2D plane. □

Step 2: Pulling Towards a Single Point

Once all spins are on a great circle, we move them to a point s perpendicular to that circle. First, we define the flow ϕ that does that. We then need to show that the energy is decreasing along the flow. This is done in two parts—initially, spins that were on one side of the great circle move closer to it, and the others move away. After a short period, the spins on the "wrong" side cross the great circle, and in the second part of the motion all spins are on a single hemisphere. Claim 2 shows that if we start ε-close to a great circle, after time 2ε all points are going to be at the same side of the circle.
We would then like to show that the energy is decreasing up to time 2ε. This unfortunately is not always true—if all spins are in the hemisphere opposite to s, then initially distances grow and the energy increases. This problem, however, could be easily solved by replacing s with $-s$, which is now in the right hemisphere. In Claim 4 we show that whenever we start ε-close to the great circle, pulling the spins to s or $-s$ results in energy decrease.

Finally, Claim 5 shows that once all spins are in the same hemisphere as s, the energy is decreasing.

Definition 4.3 Fix two spins $s, s' \in \mathbb{S}^{N-1}$. We define the function $\phi_s : \mathbb{S}^{N-1} \times \mathbb{R} \to \mathbb{S}^{N-1}$ as the flow on the sphere given by the following equation in $\mathbb{R}^N$:

$$\begin{aligned}\frac{\partial}{\partial t}\phi_s(s', t) &= s - (\phi(s', t) \cdot s)\phi(s', t),\\ \phi_s(s', 0) &= s'.\end{aligned}$$

For a configuration $S = (S_1, \dots, S_L)$ we write $\phi_s(S, t) = (\phi_s(S_1, t), \dots, \phi_s(S_L, t))$.

Observation 1 *If s' is perpendicular to s, we can solve explicitly*

$$\phi_s(s', t) = \tanh(t)\, s + \frac{1}{\cosh(t)}\, s',$$

which shows that $\lim_{t\to\infty} \phi_s(s', t) = s$. This last fact is true in general—we can always choose $\tilde{s}$ and t_0 such that

$$\phi_s(s', t) = \tanh(t - t_0)\, s + \frac{1}{\cosh(t - t_0)}\, \tilde{s},$$

by taking $\tilde{s}$ perpendicular to s in the plane spanned by s and s', and determine t_0 using $\phi_s(s', 0) = s'$.

Claim 2 *For any $\varepsilon < \frac{1}{2}$, if $\left|s \cdot s'\right| < \varepsilon$ then $\phi_s(s', 2\varepsilon) \cdot s > 0$.*

Proof Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_s(s', t) \cdot s\right) = 1 - (\phi(s', t) \cdot s)^2 \geq 0,$$

so it is enough to prove that $\phi_s(s', t) \cdot s$ cannot remain in the interval $[-\varepsilon, 0]$ for all $t \in [0, 2\varepsilon]$. But assuming it does,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_s(s', t) \cdot s\right) = 1 - (\phi(s', t) \cdot s)^2 > \frac{3}{4},$$

and therefore $\phi_s(s', 2\varepsilon) \cdot s > -\varepsilon + \frac{3}{4} \cdot 2\varepsilon = \frac{\varepsilon}{2} > 0.$ □

Claim 3 *Fix $s \in \mathbb{S}^{N-1}$, and a configuration S where all spins are on a great circle perpendicular to s. If not all spins are aligned, then*

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(\phi_s(S, t))\right|_{t=0} < 0.$$

Proof Since all spins are perpendicular to s, we can write explicitly, setting $\underline{s} = (s, \dots, s) \in \Omega$,

$$\begin{aligned}\phi_s(S, t) &= \tanh(t)\, s + \frac{1}{\cosh(t)}\, S,\\ H(\phi_s(S, t)) &= \frac{1}{2}\sum_i \|\phi_s(S, t)_i - \phi_s(S, t)_{i-1}\|^2 + \text{const.}\\ &= \frac{1}{2}\frac{1}{\cosh(t)^2} H(S) + \text{const.}\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\phi_s(S,t)) = -\frac{\sinh(t)}{\cosh(t)^3}H(S)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(\phi_s(S,t)) = \frac{2\sinh(t)^2-1}{\cosh(t)^4}H(S).$$

□

In the following we use $d(\cdot,\cdot)$ to denote the geodesic distance on the sphere.

Claim 4 *Fix $s \in \mathbb{S}^{N-1}$, and a non-constant configuration S^0 where all spins are on a great circle perpendicular to s. There exists ε small enough (depending on S^0), such that for all S satisfying $d(S_i, S_i^0) < \varepsilon$ for all i, either $H(\phi_s(S,t))$ or $H(\phi_{-s}(S,t))$ (or both) is decreasing for $t \in [0, 2\varepsilon]$.*

Proof We consider two cases. The first is when $\frac{\mathrm{d}}{\mathrm{d}t}H(\phi_s(S,0)) \le 0$. Since $\|\frac{\partial}{\partial t}\phi_s(s',t)\| \le 1$, during the time interval $[0,2\varepsilon]$ all spins of S remain at a distance at most 3ε form S^0. By the last claim $\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(\phi_s(S^0,0)) < 0$, hence we may choose ε small enough, such that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(\phi_s(S',t)) < 0$ for all S' in a 3ε-neighborhood of S^0. Then $\frac{\mathrm{d}}{\mathrm{d}t}H(\phi_s(S,t)) \le 0$ for all $t \in [0,2\varepsilon]$.

In the second case $\frac{\mathrm{d}}{\mathrm{d}t}H(\phi_s(S,0)) \ge 0$, but then by symmetry of the flow $\frac{\mathrm{d}}{\mathrm{d}t}H(\phi_{-s}(S,0)) \le 0$ and we are back to the first case. □

Claim 5 *Fix $s \in \mathbb{S}^{N-1}$ and a configuration S, such that $s \cdot S_i > 0$ for all i. Then $H(\phi_s(S,t))$ is a decreasing function for $t \in [0,\infty]$.*

Proof For all i, denoting $\underline{s} = (s,\ldots,s) \in (\mathbb{S}^{N-1})^L$,

$$\begin{aligned}
&- D_i H(S)\cdot(\underline{s} - (S\cdot\underline{s})S)_i = S_{i-1}\cdot(s-(S_i\cdot s)S_i) + S_{i+1}\cdot(s-(S_i\cdot s)S_i)\\
&\quad - (S_{i-1}\cdot S_i + S_{i+1}\cdot S_i)\, S_i\cdot(s-(S_i\cdot s)S_i)\\
&\quad = S_{i-1}\cdot s - (S_i\cdot s)(S_i\cdot S_{i-1}) + S_{i+1}\cdot s - (S_i\cdot s)(S_i\cdot S_{i+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_i -D_i H(S)\cdot(\underline{s}-(S\cdot\underline{s})S)_i\\
&\quad = 2\Big(\sum_i S_i\Big)\cdot s - \Big(\sum_i (S_i\cdot S_{i-1})S_i\Big)\cdot s - \Big(\sum_i (S_i\cdot S_{i+1})S_i\Big)\cdot s\\
&\quad = \sum((1-(S_i\cdot S_{i-1}))S_i)\cdot s + \sum((1-(S_i\cdot S_{i+1}))S_i)\cdot s > 0.
\end{aligned}$$

Note that if $s\cdot S_i > 0$ then at all positive times $\phi_s(S,t) > 0$ (since spins always get closer to s). Therefore we can plug $\phi_s(S,t)$ for S in the above inequality, and obtain:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H(\phi_s(S,t)) &= DH\cdot\frac{\partial\phi_s(S,t)}{\partial t}\\
&= DH\cdot\big(\underline{s}-(\phi(S,t)\cdot\underline{s})\phi(S,t)\big) < 0.
\end{aligned}$$

□

Step 3: Combining all Parts of the Path

Definition 4.4 Fix $T > 0$, $s \in \mathbb{S}^{N-1}$, and a configuration S^0 where all spins are on a plane perpendicular to s. Let U_{T,s,S^0} be the set of configurations S, such that:

(1) If S^0 is constant, then under the deterministic flow φ, all spins $\varphi(S,T)_i$ are at distance at most $\arccos(0.99)$ from S^0.
(2) If S^0 is not constant, for $\varepsilon = \varepsilon(S^0)$ given in Claim 4,
    (a) all spins $\varphi(S,T)_i$ are at distance at most ε from S^0_i; and
    (b) $H(\phi_s(S,t))$ is decreasing for $t \in [0, 2\varepsilon]$.

Lemma 4.2 *Fix T, s, S^0 as in definition above. Then there exists a map $\Psi_{T,s,S^0} : U_{T,s,S^0} \times [0,1] \to \Omega$ such that:*

(1) $\Psi_{T,s,S^0}(S,0) = S$ and $\Psi_{T,s,S^0}(S,1)$ is in in the arctic A.
(2) $H(\Psi(S,t))$ is non-increasing for all t.

Moreover, $v[\Psi_{T,s,S^0}]$ and $\sup_{S,t} |\Delta \operatorname{Ent}_\Psi(S,t)|$ are finite.

Proof If S^0 is non-constant, take

$$\Psi_{T,s,S^0}(S,t) = \begin{cases} \varphi(S, 3T \times t) & t \in \left[0, \frac{1}{3}\right], \\ \phi_s\left(\varphi(S,T), C \times \left(t - \frac{1}{3}\right)\right) & t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ \psi_s\left(\phi_s\left(\varphi(S,T), C/3\right), t - \frac{2}{3}\right) & t \in \left[\frac{2}{3}, 1\right], \end{cases}$$

where ψ is a rotation from s to e_1 with speed $3d(s, e_1)$. C should be chosen such that $\tanh(C/3 - 2\varepsilon)$ is close enough to 1, guaranteeing that we end up in the arctic.

If S^0 is constant, we skip ϕ and take ψ to be a rotation of S^0 to e_1.

The speed $\partial_t \Psi$ can be calculated explicitly on each part of the path—in the time interval $[0, 1/3]$ it is given by $\frac{1}{3T}\|DH\|$ which is bounded on the entire Ω. During $[1/3, 2/3]$ it is bounded by $\frac{\sqrt{L}}{C}$, since each coordinate of $\partial_t \phi(S,t)$ is bounded by 1, hence $\|\partial_t \phi(S,t)\| \le \sqrt{L}$. Finally, in the last interval the speed is bounded by $3\pi\sqrt{L}$, since during time $1/3$ each coordinate crosses distance at most π.

In order to show that the entropy is bounded we use the fact that for a flow given by an equation of the type

$$\partial_t \Phi(S,t) = f(\Phi(S,t)),$$
$$\Phi(S,0) = S,$$

the entropy production is the divergence of f:

$$\Delta \operatorname{Ent}_\Phi(S,t) = \int_0^t \operatorname{div} f(\Phi(S,u))du, \tag{53}$$

see, e.g., [19, Section 8.2]. In the first part of the path $\operatorname{div} f$ is bounded by $3T \sup |\Delta H|$, in the second part by NL, and in the third part it is 0. Since composing the paths results in adding the entropies, the overall entropy is bounded. □

Proof of Theorem 1.3 Since all limit points of the deterministic dynamics are on a great circle, any configuration S belongs to some set U_{T,s,S^0}. These are open sets, and Ω is compact, therefore there exists a finite cover $U_1, \ldots, U_K$. We set for $i = 1, \ldots, K$ the function Φ_i to be equal Ψ_{T,s,S^0}, for T, s, S^0 such that $U_i = U_{T,s,S^0}$. The hypotheses of Lemma 4.1 are then satisfied, with finite v and $\Delta \operatorname{Ent}$ (which do not depend on β), and $\Delta H \le 0$. □

4.3 Explicit Construction for $N = 3$

When $N = 3$ we are able to construct the path directly, without using a compactness argument. This enables us to obtain an explicit (though not optimal) estimate on the constant $C(L)$ in Theorem 1.3.

The path consists of three parts: first, we align the spins so that they fall close to a plane. We then contract the spins towards a well chosen pole. This second step is where the crucial difference between $O(3)$ and $O(2)$ enters. Finally, we rotate the whole configuration to get it close to e_1 and into the arctic.

The three steps are summarized in Lemmas 4.4, 4.5, and 4.6. We will need a few preparations. The first is stating that "if all spins are close to *some* great circle, they are also close to a great circle *in a fixed finite set* of great circles".

Lemma 4.3 *There exists $C > 0$ such that for any $\epsilon > 0$, one can find $K \leq C\epsilon^{-2}$ integer, and a set $\{v_1, \ldots, v_K\} \subset \mathbb{S}^2$ such that for any L, and any collection $S_1, \ldots, S_L \in \mathbb{S}^2$ satisfying $|S_i \cdot s| \leq \epsilon$ for all $i \in \{1, \ldots, L\}$ and some fixed $s \in \mathbb{S}^2$, there exits $k \in \{1, \ldots, K\}$ such that $|S_i \cdot v_k| < 2\epsilon$ for all $i \in \{1, \ldots, L\}$.*

Proof Let $\epsilon > 0$. Let $A \subset \mathbb{S}^2$ be such that for every $s \in \mathbb{S}^2$, there is $v \in A$ such that $|s - v| < \epsilon$. One can find such a set containing at most $C\epsilon^{-2}$ points with C universal (for example, by projecting on $\mathbb{S}^2$ an $\epsilon/2$ mesh-size grid on the boundary of the cube of side 2 centred at 0). Then for such a set A, for any $s \in \mathbb{S}^2$, there is $v \in A$ with $|v - s| < \epsilon$. For this v, one has $|S \cdot v| \leq |S \cdot (v - s)| + |S \cdot s| \leq \epsilon + |S \cdot s|$ for every $S \in \mathbb{S}^2$, so

$$|S \cdot s| < \epsilon \implies |S \cdot v| < 2\epsilon,$$

which gives the claim.

Fix $\sin(\frac{\pi}{16}) \geq \epsilon > 0$, and let $V_\epsilon = \{v_1, \ldots, v_K\} \subset \mathbb{S}^2$ be a set whose existence is guaranteed by the previous Lemma (with $K \leq C\epsilon^{-2}$).

For $s \in \mathbb{S}^2$, define then

$$D_1(\epsilon, s) = \{S \in \Omega_L : \ |S_i \cdot s| < \epsilon, \ \forall i = 1, \ldots, L\}.$$

Define also

$$\Omega_L^{\pm}(s, \epsilon) = \{S \in \Omega_L : \ S_i \cdot \pm s \geq 1 - 2\arcsin(\epsilon)^2 \,, \ \forall i = 1, \ldots, L\}.$$

The three steps are then represented by the following three lemmas. For the first two of them, the precise construction and proof is given in the next subsections.

Lemma 4.4 *The application $\varphi : \Omega_L \times [0, \tau] \to \Omega_L$ with $\tau = L - 1 - \epsilon$, to be constructed in Section 4.4, is such that*

(1) $\varphi(S, 0) = S$ and $\varphi(S, \tau) \in D_1(\epsilon, v)$ for some $v \in V_\epsilon$;
(2) $t \mapsto \varphi(S, t)$ is continuous, and piecewise differentiable;
(3) $t \mapsto H_L^{\mathrm{per}}(\varphi(S, t))$ is non-increasing;
(4) $\|\dot{\varphi}(S, t)\|^2 \leq 4\pi^2 L$;
(5) letting $f_t(S) = \varphi(S, t)$, one has $\left|\frac{d(f_t)_\# \nu_L}{d\nu_L}\right| \leq \left(\frac{L}{\epsilon}\right)^L$ for every $t \in [0, \tau]$ and S in the image of f_t.

Lemma 4.5 *Let $s \in \mathbb{S}^2$. The application $\phi_s : D_1(\epsilon, s) \times [0, \pi/2 - \arcsin(\epsilon)/2] \to \Omega_L$, to be constructed in Section 4.5, is such that*

(1) $\phi_s(S, 0) = S$ *and* $\phi_s(S, \pi/2 - 2\arcsin(\epsilon)) \in \Omega_L^+(s, \epsilon) \cup \Omega_L^-(s, \epsilon)$*;*
(2) $t \mapsto \phi_s(S, t)$ *is continuous, and differentiable;*
(3) $t \mapsto H_L^{\mathrm{per}}(\phi_s(S, t))$ *is non-increasing;*
(4) $\|\dot{\phi}_s(S, t)\|^2 \leq 4\pi^2 L$*;*
(5) *letting* $f_t(S) = \phi_s(S, t)$*, one has* $\left|\frac{d(f_t)_\# \nu_L}{d\nu_L}\right| \leq 2^L$*, for all* $t \in [0, \pi/2 - 2\arcsin(\epsilon)]$ *and* S *in the image of* f_t*.*

Lemma 4.6 *Let* $s \in \mathbb{S}^2$*. There exist* $\tau = \tau(s) \in [0, \pi]$*, and* $\psi_s : \Omega_L^+(s, \epsilon) \times [0, \tau] \to \Omega_L$ *such that*

(1) $\psi_s(S, 0) = S$ *and* $\psi_s(S, \tau) \in \Omega_L^+(\mathrm{e}_1, \epsilon)$*;*
(2) $t \mapsto \psi_s(S, t)$ *is continuous, and differentiable;*
(3) $t \mapsto H_L^{\mathrm{per}}(\psi_s(S, t))$ *is constant;*
(4) $\|\dot{\psi}_s(S, t)\|^2 \leq 4\pi^2 L$*;*
(5) *letting* $f_t(S) = \psi_s(S, t)$*, one has* $\left|\frac{d(f_t)_\# \nu_L}{d\nu_L}\right| = 1$ *for every* $t \in [0, \tau]$ *and* S *in the image of* f_t*.*

Proof of Lemma 4.6 Simply set τ to be the angle between s and e_1, R_t the rotation of angle t in the plane spanned by s, e_1 (with positive direction from s to e_1). Setting $\psi(S, t) = R_t(S)$ does the job. □

In the next sections, we prove Lemma 4.4 and 4.5. We then use them together with Lemma 4.1 to bound the spectral gap in Section 4.6.

4.4 Local Alignment: Proof of Lemma 4.4

The path we will use will align S_1, S_2, S_3 by rotating S_2 around the axis spanned by S_1 to end on the geodesic between S_1 and S_3, then align $S_1, \ldots, S_4$ by rotating the pair S_2, S_3 around the axis spanned by S_1 so that S_3 ends on the geodesic between S_1 and S_4, and so on and so forth. We will first consider a sequence of mappings which is better expressed in a suitable choice of coordinates. For $1 < k \leq L + 1$, consider the following coordinate system. Let $v_1 = S_1$. If $S_k \notin \{v_1, -v_1\}$, let

$$v_2 = \frac{S_k - (S_k \cdot v_1)v_1}{\sqrt{1 - (S_k \cdot v_1)^2}},$$

otherwise, let v_2 be any (fixed) norm one vector orthogonal to v_1. Let then v_3 be any vector such that (v_1, v_2, v_3) is an orthonormal basis. Express then a point s on the sphere as

$$s = (s \cdot v_1)v_1 + \sqrt{1 - (s \cdot v_1)^2}\cos(\theta)v_2 + \sqrt{1 - (s \cdot v_1)^2}\sin(\theta)v_3$$

with $\theta \in [-\pi, \pi)$ the angle between v_2 and the projection of s in the (v_2, v_3)-plane (so that $\theta = 0$ when $s = S_k$). Write $u_i = S_i \cdot v_1 \in [-1, 1]$, and θ_i for the angle θ corresponding to $s = S_i$. With this choice of coordinates, one has

$$\sum_{i=1}^{k-1} S_i \cdot S_{i+1} = \sum_{i=1}^{k-1} \left(u_i u_{i+1} + \sqrt{1 - u_i^2}\sqrt{1 - u_{i+1}^2}\cos(\theta_i - \theta_{i+1})\right),$$

where $\theta_k = 0$.

Define the path to be

$$\big(\varphi_k(S,t)\big)_i = \begin{cases} S_i & \text{if } k \le i \le L+1, \\ u_i v_1 + \sqrt{1-u_i^2}\cos(\theta_i - t\theta_{k-1})v_2 + \sqrt{1-u_i^2}\sin(\theta_i - t\theta_{k-1})v_3 & \text{if } 1 < i < k. \end{cases}$$

As, in the chosen coordinates, $d\nu_L(S) = d\nu(S_1)\prod_{i=k}^{L} d\nu(S_i)\prod_{i=2}^{k-1} du_i d\theta_i$, one has

$$\Big|\frac{d(f_t)_\# \nu_L}{d\nu_L}\Big| = \frac{1}{1-t},$$

where $f_t(S) = \varphi_k(S,t)$.

Note then that

$$\begin{aligned} -H_L^{\mathrm{per}}(\varphi_k(S,t)) &= \sum_{i=1}^{k-1}\big(\varphi_k(S,t)\big)_i \cdot \big(\varphi_k(S,t)\big)_{i+1} + \sum_{i=k}^{L} S_i \cdot S_{i+1} \\ &= \sum_{i=1}^{k-2} S_i \cdot S_{i+1} + \Big(u_{k-1}u_k + \sqrt{1-u_{k-1}^2}\sqrt{1-u_k^2}\cos((1-t)\theta_{k-1})\Big) \\ &\quad + \sum_{i=k}^{L} S_i \cdot S_{i+1}. \end{aligned}$$

As the angles are in $[-\pi,\pi]$, $-H_L^{\mathrm{per}}(\varphi_k(S,t)) \ge -H_L^{\mathrm{per}}(S)$.

One has that $\big(\varphi_k(S,1)\big)_{k-1}$ is in the plane spanned by S_1 and S_k, and that the distance to that plane of $\big(\varphi_k(S,1-\epsilon)\big)_{k-1}$ is at most

$$|\big(\varphi_S^{(k)}(1-\epsilon)\big)_{k-1} \cdot v_3| \le \sin(\epsilon\pi).$$

Let $\tau_0 = 0$, $\tau_k = \tau_{k-1} + 1 - \epsilon/L$. Define

$$\begin{gathered} \varphi(S,0) = S, \\ \varphi(S,t) = \varphi_k(\varphi(S,\tau_{k-1}), t-\tau_{k-1}),\ t \in [\tau_{k-1},\tau_k]. \end{gathered}$$

This defines a continuous, and piecewise differentiable function of $t \in [0,\tau_{L-1}]$, where $\tau_{L-1} = L-1-\epsilon$. By the triangle inequality, the distance of $(\varphi(S,\tau_{L-1}))_i$ to the plane spanned by S_1 and S_L is at most ϵ. By the previous observations, the Hamiltonian is decreasing in t and

$$\Big|\frac{d(f_t)_\# \nu_L}{d\nu_L}\Big| \le (L\epsilon^{-1})^L,$$

where $f_t(S) = \varphi(S,t)$.

It remains to prove the bound on $\|\dot\varphi(S,t)\|^2$. On each time interval $[\tau_{k-1},\tau_k]$, the mapping is a rotation of the k first spins around the axis spanned by S_1, so $\|\dot\varphi(S,t)\|^2 \le 4\pi^4 L$.

4.5 Contraction Towards a Pole: Proof of Lemma 4.5

For $s \in \mathbb{S}^2$, parametrize $S \in \Omega_L$ by $(\theta,\theta') \in ([-\frac{\pi}{2},\frac{\pi}{2}]\times[0,2\pi])^L$ via $S_i = \cos(\frac{\pi}{2}+\theta_i)s + \sin(\frac{\pi}{2}+\theta_i)\cos(\theta_i')v_1 + \sin(\frac{\pi}{2}+\theta_i)\sin(\theta_i')v_2$ where (s,v_1,v_2) form an orthonormal basis of $\mathbb{R}^3$. Define the configuration $X_t = X_t(S) = X_t(\theta,\theta')$ via

$$\big(X_t(\theta,\theta')\big)_i = \cos(\frac{\pi}{2}+\theta_i+t)s + \sin(\frac{\pi}{2}+\theta_i+t)\cos(\theta_i')v_1 + \sin(\frac{\pi}{2}+\theta_i+t)\sin(\theta_i')v_2$$

Using $\sin(\frac{\pi}{2}+a)=\cos(a)$ and $\cos(\frac{\pi}{2}+a)=-\sin(a)$, one obtains

$$-H_L^{\mathrm{per}}(X_t(\theta,\theta'))=\sum_{i=1}^{L}\sin(\theta_i+t)\sin(\theta_{i+1}+t)+\cos(\theta_i+t)\cos(\theta_{i+1}+t)\cos(\theta_i'-\theta_{i+1}').$$

So, as $\sin a\cos b+\sin b\cos a=\sin(a+b)$,

$$-\frac{d}{dt}H_L^{\mathrm{per}}(X_t(\theta,\theta'))=\sum_{i=1}^{L}\sin(\theta_i+\theta_{i+1}+2t)(1-\cos(\theta_i'-\theta_{i+1}')).$$

Moreover,

$$-\frac{d^2}{dt^2}H_L^{\mathrm{per}}(X_t(\theta,\theta'))=2\sum_{i=1}^{L}\cos(\theta_i+\theta_{i+1}+2t)(1-\cos(\theta_i'-\theta_{i+1}')).$$

First observe three things:

(1) if $0\le\theta_i+t\le\pi/2$ for all is, then $-\frac{d}{dt}H_L^{\mathrm{per}}(X_t(\theta,\theta'))\ge 0$;
(2) if $-\pi/2\le\theta_i-t\le 0$ for all is, then $-\frac{d}{dt}H_L^{\mathrm{per}}(X_{-t}(\theta,\theta'))\ge 0$;
(3) if $|\theta_i|<\frac{\pi}{8}$ for all is and $t<\frac{\pi}{8}$, $\cos(\theta_i+\theta_{i+1}+2t)\ge 0$ for all is and so $-\frac{d^2}{dt^2}H_L^{\mathrm{per}}(X_t(\theta,\theta'))\ge 0$.

When $S\in D_1(\epsilon,s)$, we have $|\theta_i|\le\arcsin(\epsilon)$. Therefore, by our constraint on ϵ, the sign of $-\frac{d}{dt}H_L^{\mathrm{per}}(X_t(\theta,\theta'))$ at $t=0$ determines the desired monotonicity for either $t\in[0,\pi/2-2\arcsin(\epsilon)]$ or $-t\in[0,\pi/2-2\arcsin(\epsilon)]$: indeed, the third observation above gives the monotonicity for all $|t|\le\pi/8$, and the first two observations show the monotonicity for $|t|>\pi/8$ since $|\theta_i|\le\arcsin(\epsilon)$.

We can now construct the path ϕ_s as follows:

- If $S\in D_1(\epsilon,s)$ is such that $-\frac{d}{dt}H_L^{\mathrm{per}}(X_t(S))\big|_{t=0}\ge 0$, set
$$\phi_s(S,t)=X_t(S),$$
for $t\in[0,\pi/2-2\arcsin(\epsilon)]$.
- If $S\in D_1(\epsilon,s)$ is such that $-\frac{d}{dt}H_L^{\mathrm{per}}(X_t(S))\big|_{t=0}<0$, set
$$\phi_s(S,t)=X_{-t}(S),$$
for $t\in[0,\pi/2-2\arcsin(\epsilon)]$.

As $\epsilon<\sin(\frac{\pi}{16})$, the first three points of Lemma 4.5 are obvious (the first one follows from $(\phi_s(S,t))_i\cdot s=\cos(\pi/2+\theta_i\pm t)$ depending on the case). To obtain the one-before-last, observe that the transformation is a rotation of each coordinates, so $\|\dot{\phi}_s(S,t)\|^2\le 4\pi^2L$.

The last point of Lemma 4.5 follows from

$$d\nu_L(S)=\prod_{i=1}^{L}\cos(\theta_i)d\theta_i d\theta_i',$$

and $\frac{\cos(\theta_i-t)}{\cos(\theta_i)}\le\frac{1}{\cos(\pi/16)}\le 2$ for θ_i,t in the concerned regions.

4.6 Spectral Gap

Let $\epsilon > 0$. Rather than parametrizing the sets U_i's of Lemma 4.1 with numbers, we label them with pairs (σ, v) with $\sigma \in \{-, +\}$ and $v \in V_\epsilon$, where V_ϵ is given as right after Lemma 4.3. Explicitly, we use $U_{\sigma,v} = \{s : \chi_{\sigma,v}(s) = 1\}$, where, for $v \in V_\epsilon$, $\sigma \in \{-, +\}$,

$$\chi_{\sigma,v}(S) = \mathbf{1}_{\varphi(S,\tau_1)\in D_1(\epsilon,v)}\mathbf{1}_{\phi_v(\varphi(S,\tau_1),\tau_2)\in\Omega_L^\sigma(v,\epsilon)},$$

and $\tau_1 = L - 1 - \epsilon$ and φ are as in Lemma 4.4, while $\tau_2 = \frac{\pi}{2} - 2\arcsin(\epsilon)$, and ϕ_s are as in Lemma 4.5. Finally, let $\tau_3(s)$, ψ_s be as in Lemma 4.6, and define

$$T_1 = \tau_1, \quad T_2 = \tau_1 + \tau_2, \quad T_3 = \tau_1 + \tau_2 + \tau_3(\sigma v)\,.$$

Next, introduce the path

$$\Psi_{\sigma,v}(S,t) = \begin{cases} \varphi(S,t) & \text{if } t \le T_1, \\ \phi_v(\varphi(S,T_1), t - T_1) & \text{if } t \in [T_1, T_2], \\ \psi_v\big(\phi_v(\varphi(S,T_1), T_2 - T_1), t - T_2\big) & \text{if } t \in [T_2, T_3], \end{cases}$$

which is defined on $U_{\sigma,v}$, is continuous, and is piecewise differentiable in t with the bound

$$\|\dot{\Psi}_{\sigma,v}(S,t)\|^2 \le 4\pi^2 L, \tag{54}$$

at every point of differentiability. Moreover, for any L large enough

$$\left|\frac{d(f_t)_\# \nu_L}{d\nu_L}\right| \le (L\epsilon^{-1})^L, \tag{55}$$

where $f_t(S) = \Psi_{\sigma,v}(S,t)$. Finally, from the monotonicity of the Hamiltonian along each piece of the path,

$$-H_L^{\text{per}}(\Psi_{\sigma,v}(S,t)) \ge -H_L^{\text{per}}(S). \tag{56}$$

Applying Lemma 4.1 with these estimates on the functions $(S,t) \mapsto \Psi_{\sigma,v}(S, t/T_3)$, and choosing ϵ such that $\Omega_L^+(e_1,\epsilon) \subseteq A$, yields the bound on the relaxation time stated in Theorem 1.3 in the case $N = 3$. $\square$

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflicts of interest All authors have no conflicts of interest.

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